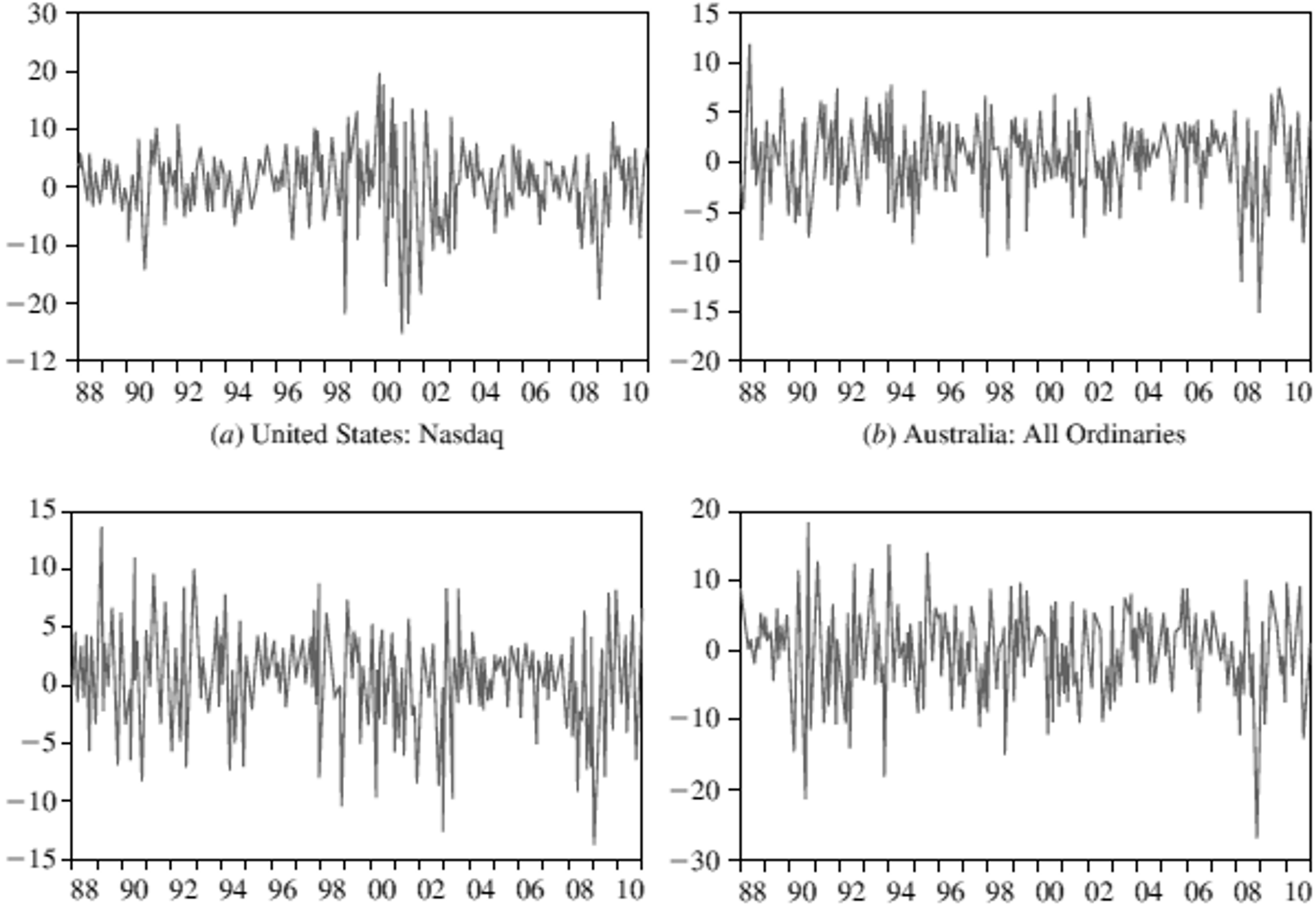
Topic Four: Time-Varying Volatility and ARCH Models

Outline

- Time-Varying Volatility
- The ARCH Model
- Testing, Estimating, and Forecasting
- Extensions
  - ARCH (q)
  - GARCH
  - T-GARCH
  - GARCH in Mean
  - … …

Time-Varying Volatility



(*a*) United States: Nasdaq

(*b*) Australia: All Ordinaries

(*c*) United Kingdom: FTSE

(*d*) Japan: Nikkei

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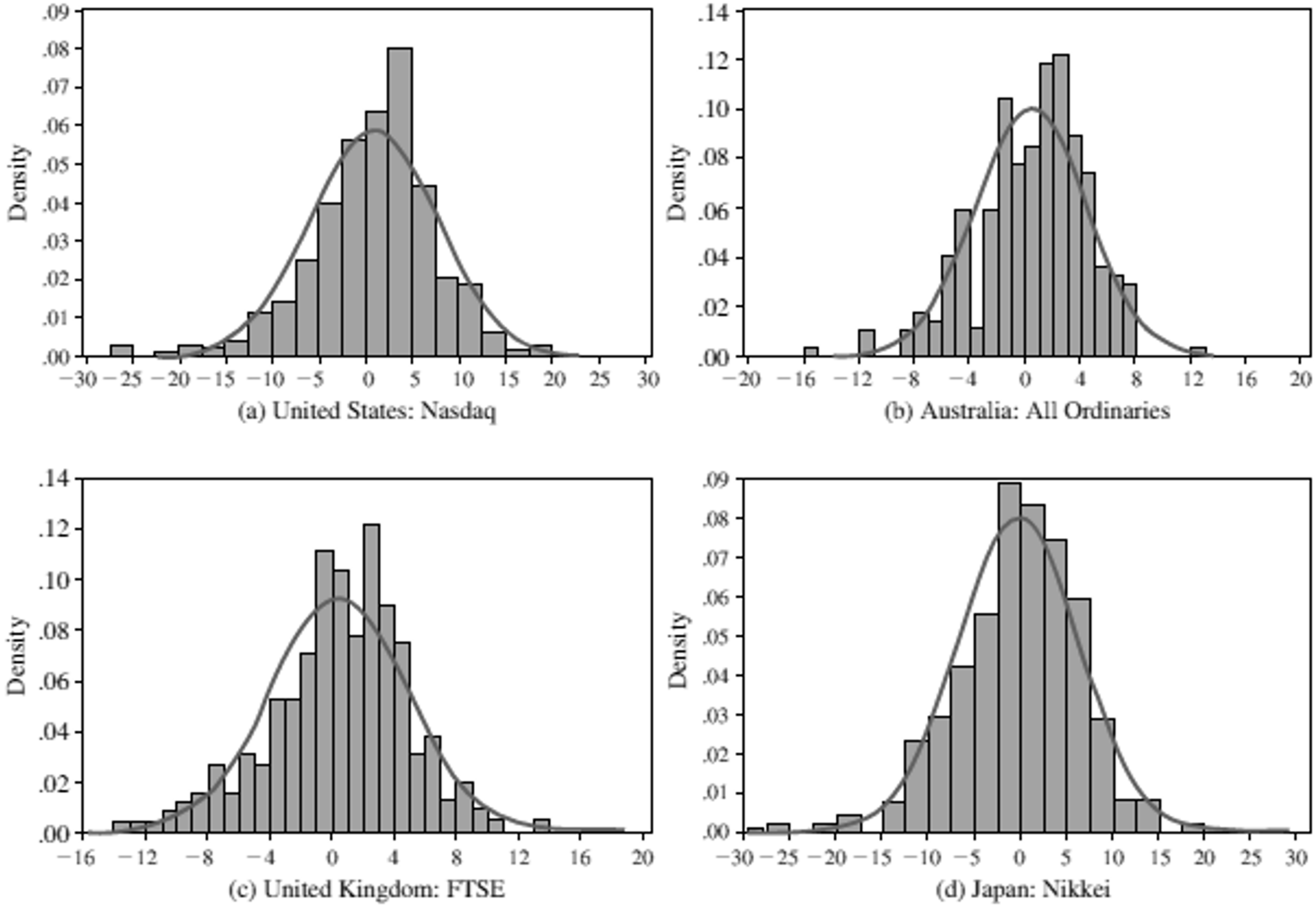
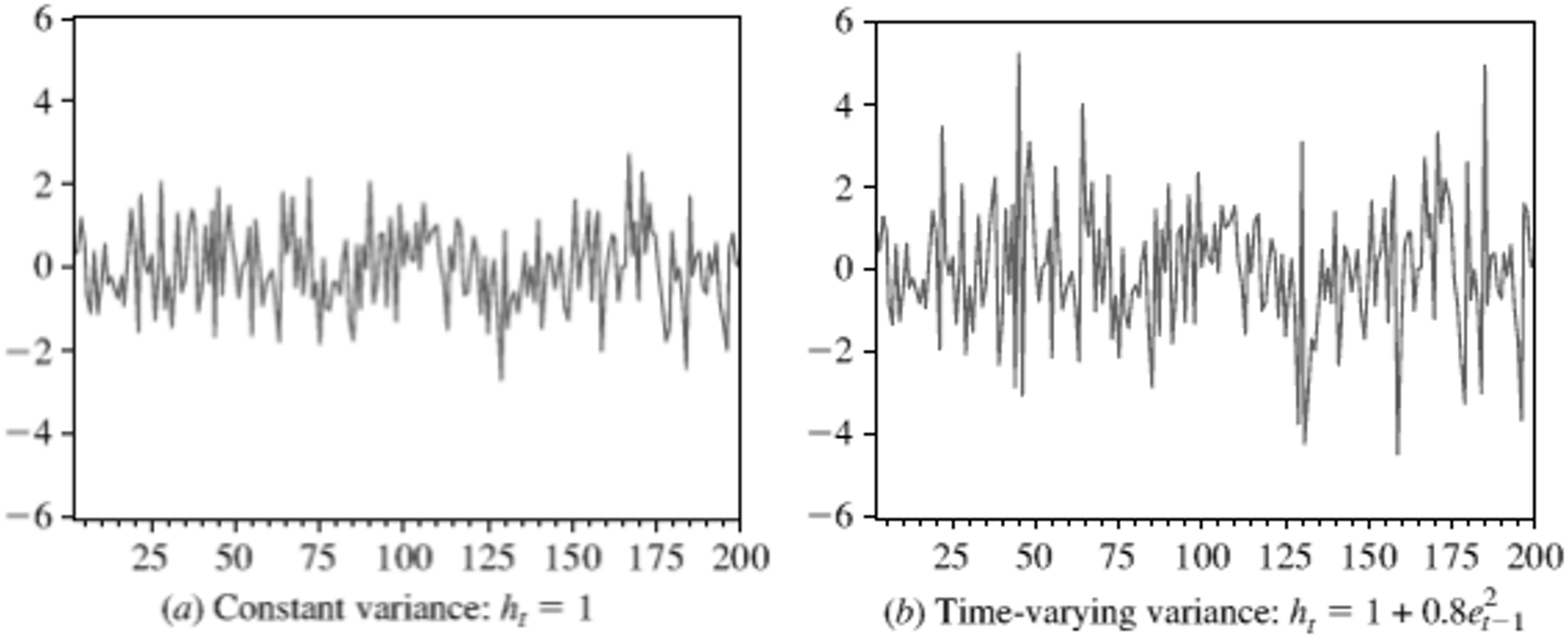
Time-Varying Volatility



Time-Varying Volatility



Simulated examples of constant and time-varying variances

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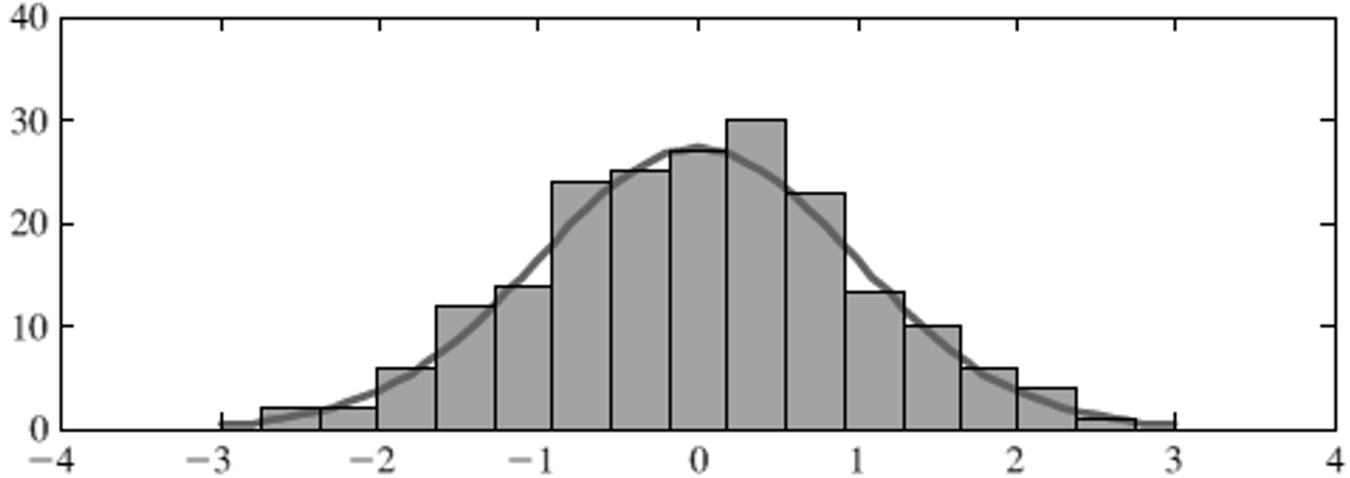
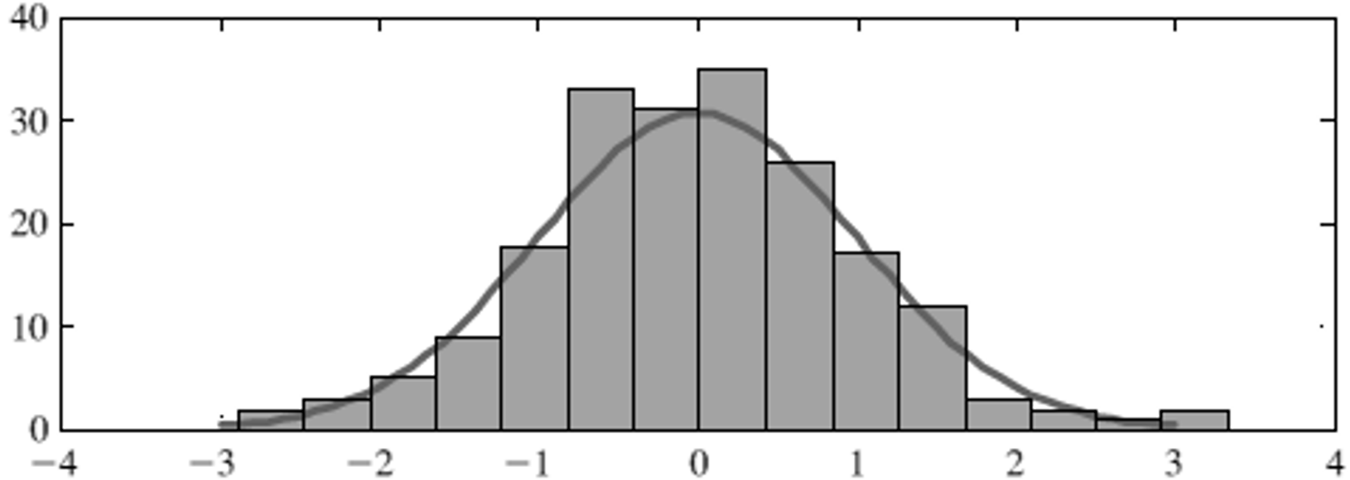
Time-Varying Volatility



(a) Constant variance



(b) Time-varying variance

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The ARCH Model

- Autoregressive conditional heteroskedastic (ARCH) model
- Deals with stationary series, but with conditional variances that change over time
- Consider a model with an AR(1) error term:

$$y_t = \phi + e_t, \quad e_t = \rho e_{t-1} + v_t, \quad |\rho| < 1 \quad v_t \sim N(0, \sigma_v^2)$$

- The **unconditional mean** of the error is:

$$E[e_t] = E[v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots] = 0$$

- The **conditional mean** for the error is:

$$E[e_t \mid I_{t-1}] = E[\rho e_{t-1} \mid I_{t-1}] + E[v_t] = \rho e_{t-1}$$

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The ARCH Model

- The **unconditional variance** of the error is:

$$E\left[e_t - 0\right]^2 = E\left[v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots\right]^2$$
$$= E\left[v_t^2 + \rho^2 v_{t-1}^2 + \rho^4 v_{t-2}^2 + \cdots\right]$$
$$= \sigma_v^2\left[1 + \rho^2 + \rho^4 + \cdots\right]$$
$$= \frac{\sigma_v^2}{1-\rho^2}$$

- The **conditional variance** for the error is:

$$E\left[\left(e_t - \rho e_{t-1}\right)^2 \mid I_{t-1}\right] = E\left[v_t^2 \mid I_{t-1}\right] = \sigma_v^2$$

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The ARCH Model

- Suppose that instead of a conditional mean that changes over time we have a conditional variance that changes over time
- Consider the AR(1) model on slide 7:

$$y_t = \phi + e_t, \quad e_t \mid I_{t-1} \sim N(0, h_t)$$

$$h_t = \alpha_0 + \alpha_1 e_{t-1}^2, \quad \alpha_0 > 0, \quad 0 \leq \alpha_1 < 1$$

- ARCH: time-varying variances (heteroskedasticity) that depend on (are conditional on) lagged effects (autocorrelation)

The ARCH Model

- The ARCH model is useful for modeling volatility and especially changes in volatility over time
- The ARCH model is intuitively appealing because it seems sensible to explain volatility as a function of the errors e_t
  - These errors are often called "shocks" or "news" by financial analysts
    - They represent the unexpected!
  - According to the ARCH model, the larger the shock, the greater the volatility in the series
  - This model captures volatility clustering, as big changes in e_t are fed into further big changes in h_t via the lagged effect e_{t-1}

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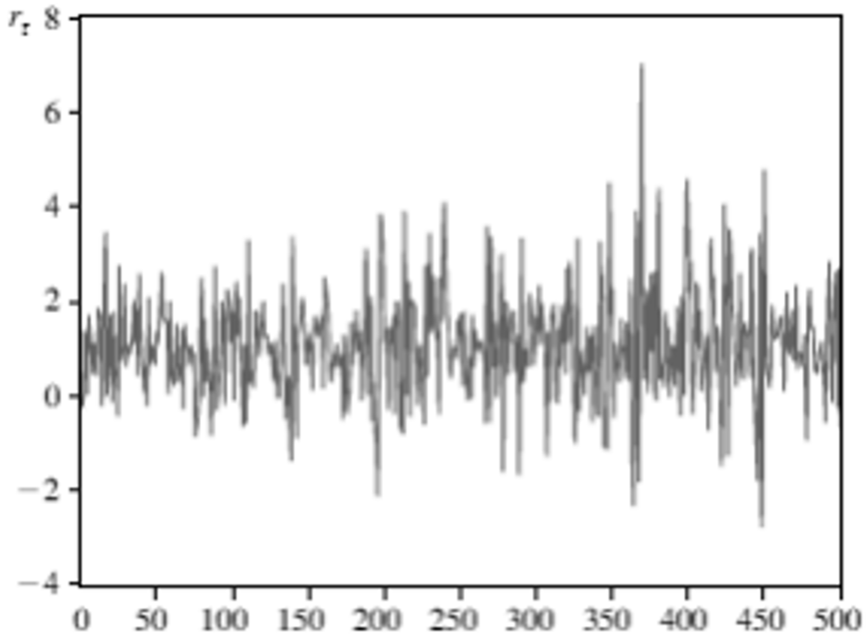
Testing, Estimating, and Forecasting

- Testing: Lagrange multiplier (LM) test for ARCH
- Test equation: $\hat{e}_t^2 = \gamma_0 + \gamma_1 \hat{e}_{t-1}^2 + v_t$
- Hypotheses: $H_0 : \gamma_1 = 0 \qquad H_1 : \gamma_1 \neq 0$
- Example: Shares of Brighten Your Day (BYD) Lighting



Time series of returns for BYD Lighting

Testing, Estimating, and Forecasting

- The results for an ARCH test are:

$$\hat{e}_t^2 = 0.908 + 0.353\hat{e}_{t-1}^2 \qquad R^2 = 0.124$$
$$(t) \qquad\qquad (8.409)$$

  - The t-statistic suggests a significant first-order coefficient
  - The sample size is 500, giving LM test value of $(T - q)R^2 = 62.16$
  - Comparing the computed test value to the 5% critical value of a $\chi^2_{(1)}$ distribution ($\chi^2_{(0.95, 1)} = 3.841$) leads to the rejection of the null hypothesis
  - The residuals show the presence of ARCH(1) effects.

Testing, Estimating, and Forecasting

- Estimating: maximum likelihood method
- Example: Brighten Your Day (BYD) Lighting
- The estimated models are:

$$\hat{r}_t = \hat{\beta}_0 = 1.063$$

$$\hat{h}_t = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{e}_{t-1}^2 = 0.642 + 0.569 \hat{e}_{t-1}^2$$

$$(t) \qquad\qquad\qquad\qquad (6.877)$$

- The forecast return and volatility are:

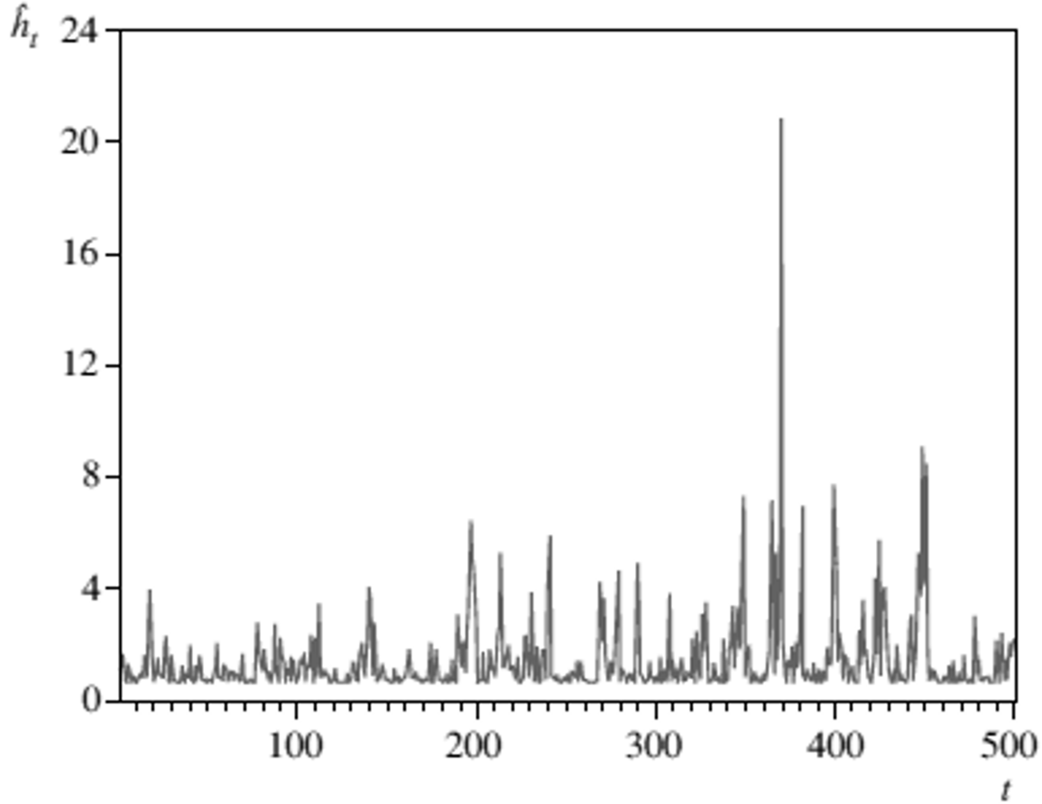
$$\hat{r}_{t+1} = \hat{\beta}_0 = 1.063$$

$$\hat{h}_{t+1} = \hat{\alpha}_0 + \hat{\alpha}_1 \left(r_t - \hat{\beta}_0 \right)^2 = 0.642 + 0.569 \left(r_t - 1.063 \right)^2$$

Testing, Estimating, and Forecasting

- The forecast return and volatility are:

$$\hat{r}_{t+1} = \hat{\beta}_0 = 1.063$$
$$\hat{h}_{t+1} = \hat{\alpha}_0 + \hat{\alpha}_1\left(r_t - \hat{\beta}_0\right)^2 = 0.642 + 0.569\left(r_t - 1.063\right)^2$$



Extensions

- From ARCH(1) to ARCH(q):
- Functional form:

$$h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 \ldots + \alpha_q e_{t-q}^2$$

- Testing, estimating, and forecasting, are natural extensions of the case with one lag
- One of the shortcomings of an ARCH(q) model is that there are q + 1 parameters to estimate, which may lose accuracy in the estimation
- The generalized ARCH model, or GARCH, is an alternative way to capture long lagged effects with fewer parameters

Extensions

- From ARCH(q) to GARCH (1, 1) model
- Re-write $h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 ... + \alpha_q e_{t-q}^2$ as:

$$h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \alpha_1 e_{t-2}^2 + \beta_1^2 \alpha_1 e_{t-3}^2 + \cdots$$

or

$$h_t = \left(\alpha_0 - \beta_1 \alpha_0\right) + \alpha_1 e_{t-1}^2 + \beta_1 \left(\alpha_0 + \alpha_1 e_{t-2}^2 + \beta_1 \alpha_1 e_{t-3}^2 + \cdots\right)$$

but, since:

$$h_{t-1} = \alpha_0 + \alpha_1 e_{t-2}^2 + \beta_1 \alpha_1 e_{t-3}^2 + \beta_1^2 \alpha_1 e_{t-4}^2 + \cdots$$

we get: $h_t = \delta + \alpha_1 e_{t-1}^2 + \beta_1 h_{t-1}$

Extensions

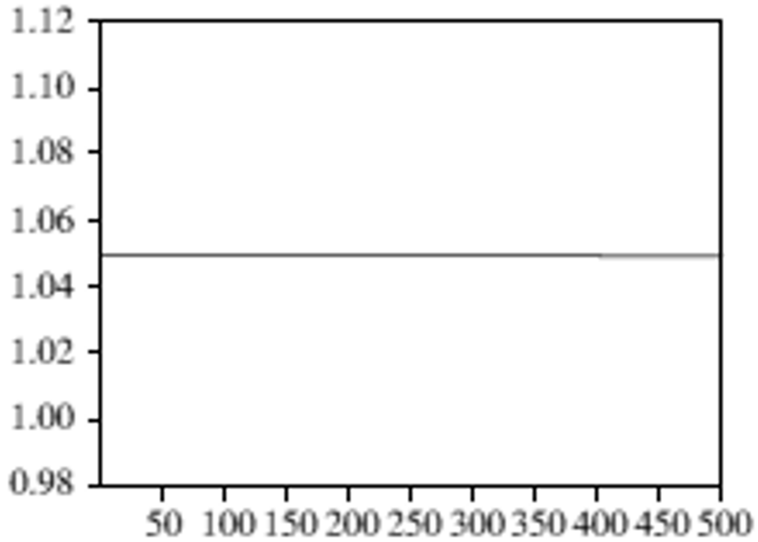
- GARCH (1, 1) model
  - The model is a very popular specification because it fits many data series well
  - It tells us that the volatility changes with lagged shocks (e^2_{t-1}) but there is also momentum in the system working via h_{t-1}
  - One reason why this model is so popular is that it can capture long lags in the shocks with only a few parameters

Extensions

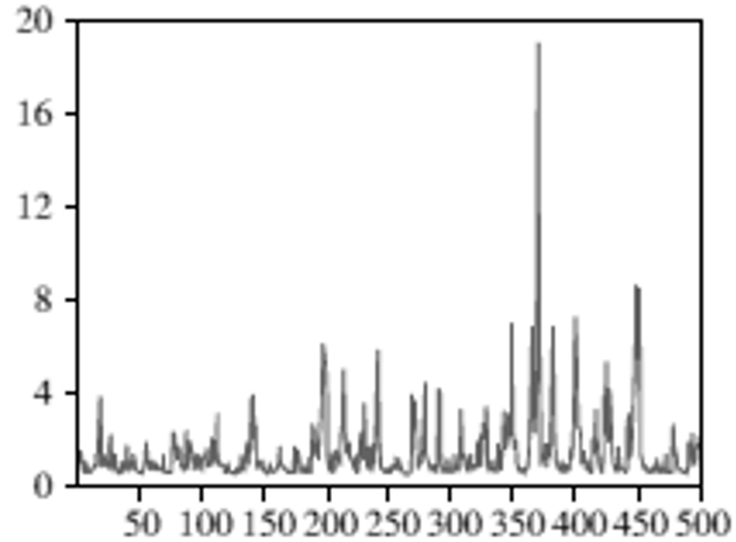
- GARCH (1, 1) model
- Example: Brighten Your Day (BYD) Lighting
- The estimated model is:

$$\hat{r}_t = 1.049 \qquad \hat{h}_t = 0.401 + 0.492\hat{e}^2_{t-1} + 0.238\hat{h}_{t-1}$$

$$(t) \qquad\qquad (4.834) \qquad (2.136)$$



(a) GARCH(1,1): $E(r_t) = 1.049$



(b) GARCH(1,1): $h_t = 0.401 + 0.492e^2_{t-1} + 0.238h_{t-1}$

Extensions

- The threshold ARCH model, or T-ARCH
- Positive and negative news are treated asymmetrically
- The specification of the conditional variance is:

$$h_t = \delta + \alpha_1 e_{t-1}^2 + \gamma d_{t-1} e_{t-1}^2 + \beta_1 h_{t-1}$$

$$d_t = \begin{cases} 1 & e_t < 0 \text{ (bad news)} \\ 0 & e_t \geq 0 \text{ (good news)} \end{cases}$$

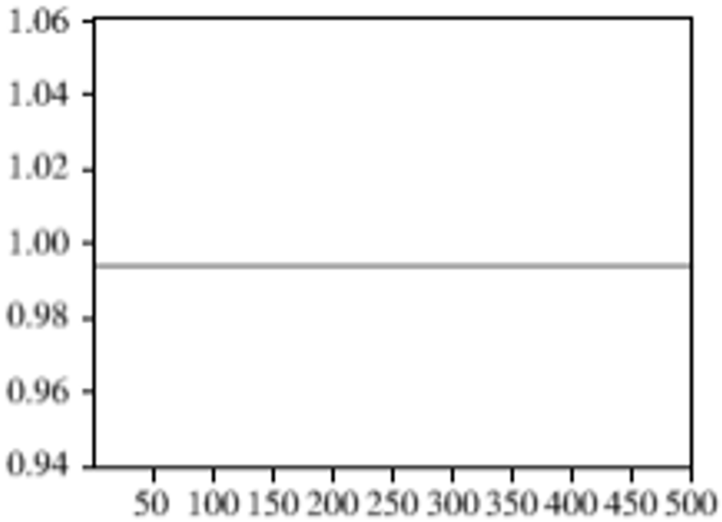
Extensions

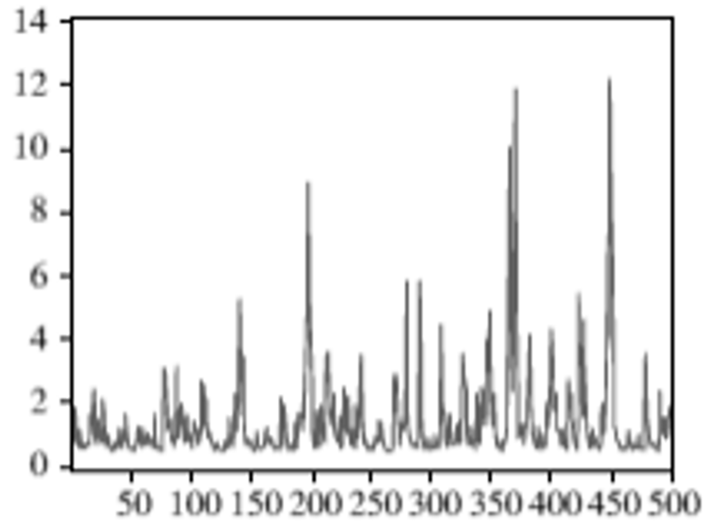
- The threshold ARCH model, or T-ARCH
- Example: Brighten Your Day (BYD) Lighting
- The estimated model is:

$$\hat{r}_t = 0.994$$

$$\hat{h}_t = 0.356 + 0.263\hat{e}^2_{t-1} + 0.492 d_{t-1}\hat{e}^2_{t-1} + 0.287\hat{h}_{t-1}$$

$$(t) \qquad (3.267) \qquad (2.405) \qquad (2.488)$$


(c) T-GARCH(1,1): $E(r_t) = 0.994$


(d) T-GARCH(1,1): $h_t = 0.356 + (0.263 + 0.492 d_{t-1})e^2_{t-1} + 0.287 h_{t-1}$

Extensions

- GARCH-in-mean model
- Captures the relationship between risk and return (variance and mean)
- The functional form is:

$$y_t = \beta_0 + \theta h_t + e_t$$

$$e_t \mid I_{t-1} \sim N(0, h_t)$$

$$h_t = \delta + \alpha_1 e_{t-1}^2 + \beta_1 h_{t-1},$$

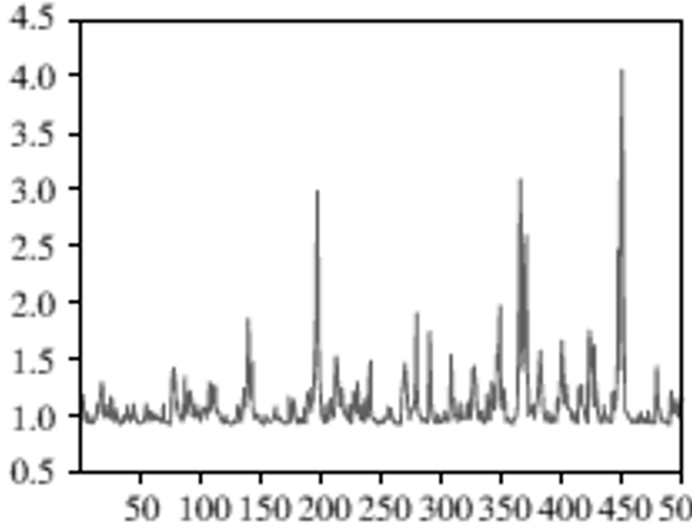
$$\delta > 0,\ 0 \le \alpha_1 < 1,\ 0 \le \beta_1 < 1$$

Extensions

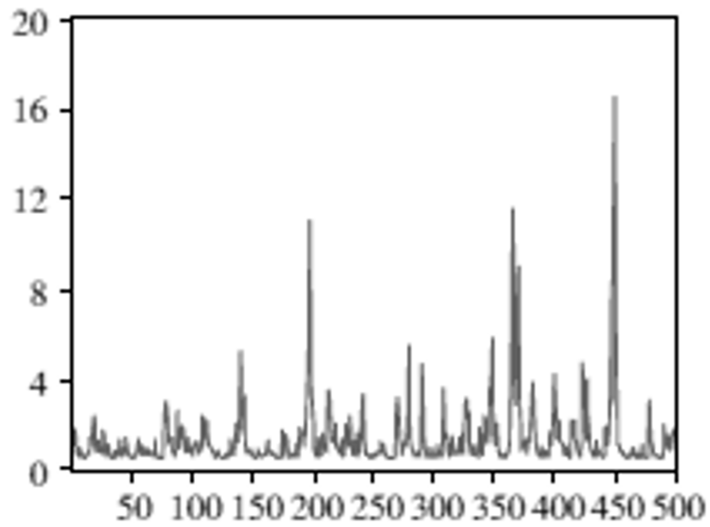
- GARCH-in-mean model
- Example: Brighten Your Day (BYD) Lighting
- The estimated model is:

$$\hat{r}_t = 0.818 + 0.196 h_t$$
$$(t) \qquad (2.915)$$

$$\hat{h}_t = 0.370 + 0.295\hat{e}^2_{t-1} + 0.321 d_{t-1}\hat{e}^2_{t-1} + 0.278\hat{h}_{t-1}$$
$$(t) \qquad (3.426) \qquad (1.979) \qquad (2.678)$$



(*e*) GARCH-in-mean: $E(r_t) = 0.818 + 0.196h_t$



(*f*) GARCH-in-mean:
$h_t = 0.370 + (0.295 + 0.321d_{t-1})e^2_{t-1} + 0.278h_{t-1}$

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Summary

- Time-Varying Volatility
- The ARCH Model
- Testing, Estimating, and Forecasting
- Extensions
    - ARCH (q)
    - GARCH
    - T-GARCH
    - GARCH in Mean