Topic Three
Vector Error Correction and Vector Autoregressive Models

Outline

- VEC and VAR Models
- Estimating a Vector Error Correction Model
- Estimating a VAR Model
- Impulse Responses and Variance Decomposition

VEC and VAR Models

- Some jargons:
- In $y_t = \beta_{10} + \beta_{11} x_t + e_t^y, \quad e_t^y \sim N(0, \sigma_y^2)$ we normalized on y.
- In $x_t = \beta_{20} + \beta_{21} y_t + e_t^x, \quad e_t^x \sim N(0, \sigma_x^2)$ we normalized on x.
- Univariate analysis examines a single data series
- Bivariate analysis examines a pair of series
- Vector analysis examines a number of series: one, two, or more
- Vector analysis is a generalization of the univariate and bivariate analyses

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VEC and VAR Models

- Consider a vector autoregression (VAR) model:

$$y_t = \beta_{10} + \beta_{11} y_{t-1} + \beta_{12} x_{t-1} + v_t^y$$

$$x_t = \beta_{20} + \beta_{21} y_{t-1} + \beta_{22} x_{t-1} + v_t^x$$

- Since the maximum lag is of order 1, we have a VAR(1)
- If y and x are I(1) and not cointegrated, estimate the following model simultaneously

$$\Delta y_t = \beta_{11} \Delta y_{t-1} + \beta_{12} \Delta x_{t-1} + v_t^{\Delta y}$$

$$\Delta x_t = \beta_{21} \Delta y_{t-1} + \beta_{22} \Delta x_{t-1} + v_t^{\Delta x}$$

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VEC and VAR Models

- If y and x are I(1) and cointegrated, we can estimate a vector error correction (VEC) model
- For example, y_t and x_t are integrated of order 1 so that $y_t = \beta_0 + \beta_1 x_t + e_t$
- The VEC model is:

$$\Delta y_t = \alpha_{10} + \alpha_{11}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^y$$

$$\Delta x_t = \alpha_{20} + \alpha_{21}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^x$$

- The coefficients α_{11}, α_{21} are known as error correction coefficients. α_{11} should be negative, and the rest of the EC coefficients should be positive

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Estimating a VEC Model

- A two step least squares procedure:
  - Use least squares to estimate the cointegrating relationship and generate the lagged residuals

    $$\hat{e}_t = y_t - b_0 - b_1 x_t$$

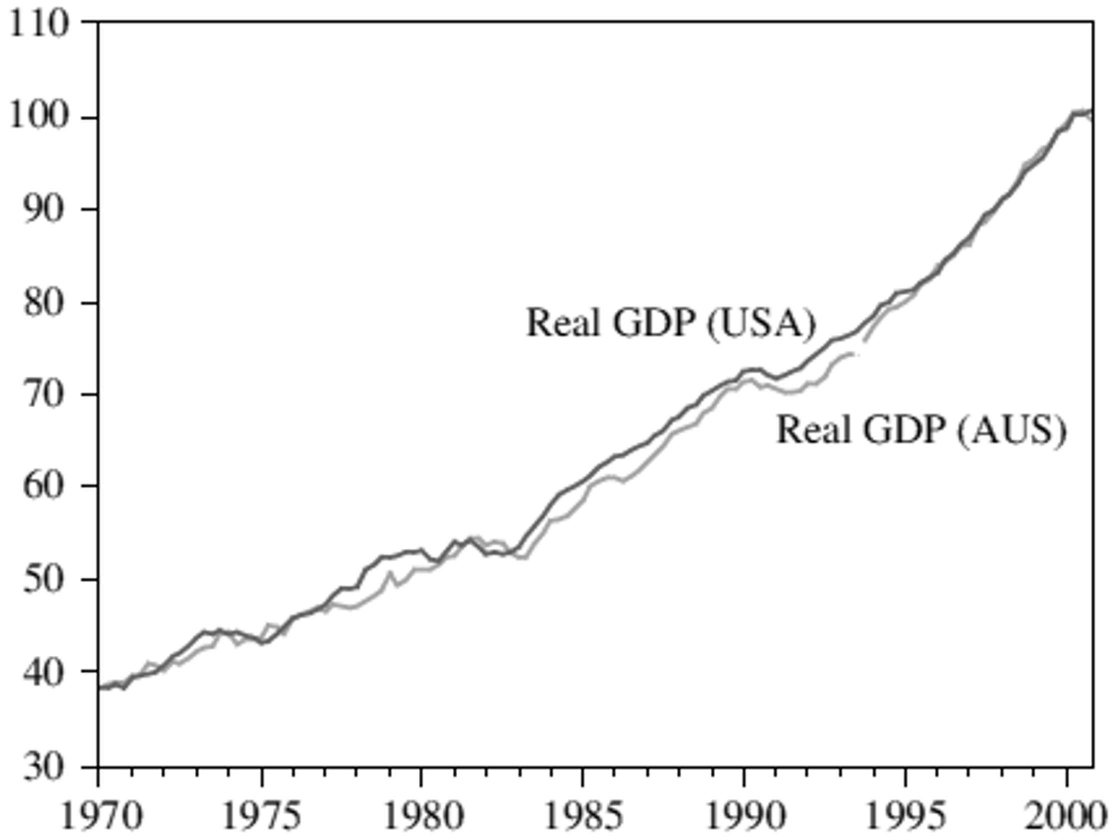
  - Use least squares to estimate the equations:

    $$\Delta y_t = \alpha_{10} + \alpha_{11}\hat{e}_{t-1} + v_t^y$$

    $$\Delta x_t = \alpha_{20} + \alpha_{21}\hat{e}_{t-1} + v_t^x$$

Estimating a VEC Model

- Example: Quarterly real GDP of a small economy (Australia) and a large economy (US)



Real gross domestic products (GDP = 100 in 2000)

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Estimating a VEC Model

- Example: Quarterly real GDP of Australia & US
- Step 1: check for cointegration (the intercept term is omitted because it has no economic meaning):

$$\hat{A}_t = 0.985 U_t$$

- The Unit Root test result is:

$$\widehat{\Delta \hat{e}_t} = -.128 \hat{e}_{t-1}$$

$$(tau) \quad (-2.889)$$

- As the critical value is -2.76, reject the null hypothesis. The two series are cointegrated.

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Estimating a VEC Model

- Example: Quarterly real GDP of Australia & US
- Step 2: estimate a VEC model for $\{A_t, U_t\}$:

$$\widehat{\Delta A}_t = -0.0646 - 0.1280\hat{e}_{t-1} - 0.0021\Delta A_{t-1} + 0.2064\Delta U_{t-1}$$

(t) (-3.52) (-0.02) (1.63)

$$\widehat{\Delta U}_t = 0.2266 - 0.0350\hat{e}_{t-1} + 0.0791\Delta A_{t-1} + 0.1040\Delta U_{t-1}$$

(t) (-1.17) (0.82) (2.19)

- Note that the first EC coefficient is negative and significant. The quarterly adjustment of A_t is about 10% of the deviation of A_{t-1} from its cointegrating value $0.985U_{t-1}$.
- The second EC coefficient is insignificant.
- Small economy is likely to react to economic conditions in the large economy, but not vice versa.

Estimating a VAR Model

- A two step least squares procedure:
  - Transform non-stationary variables to stationary variables by first differencing

    $\Delta y_t = y_t - y_{t-1}, \quad \Delta x_t = x_t - x_{t-1}$
  - Estimate a VAR model for the set of I(0) variables derived above.

Estimating a VAR Model

- Example: Real personal disposable income (Y) and real personal consumption expenditure (C)
- ADF test for Y and C: both are I(1)

Variable	Test statistic	P-value
Y	-2.741	0.068
C	-1.995	0.289
ΔY	-3.529	<0.001
ΔC	-5.045	<0.001

- Check for cointegration: not cointegrated

$$\hat{e}_t = C_t + 0.404 - 1.035 Y_t$$

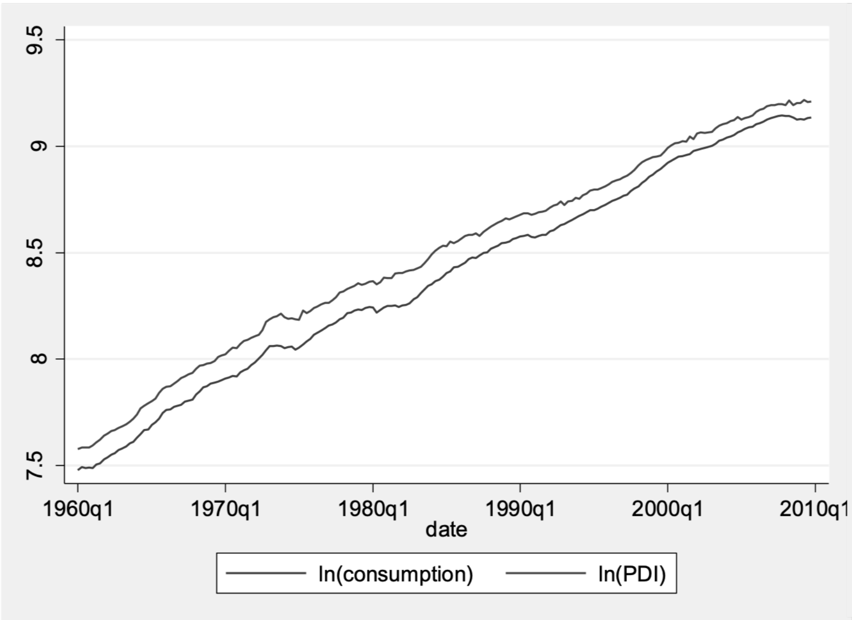
$$\Delta\hat{e}_t = -0.088\hat{e}_{t-1} - 0.299\Delta\hat{e}_{t-1}$$

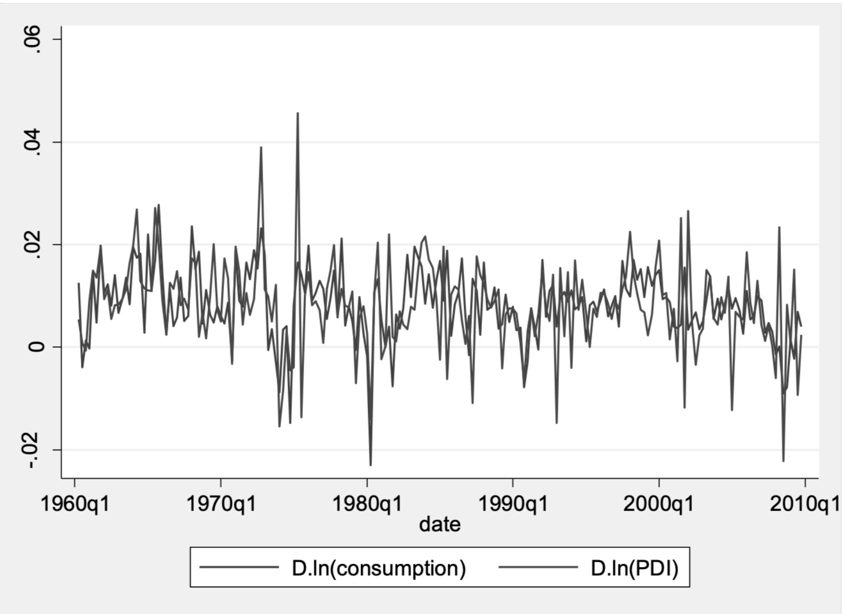
$$(tau) \quad (-2.873)$$

Critical value: -3.37

Estimating a VAR Model

- Example: Real personal disposable income (Y) and real personal consumption expenditure (C)





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Estimating a VAR Model

- Use LM test to select lag length. The VAR model is estimated as:

$$\Delta \hat{C}_t = 0.005 + 0.215 \Delta C_{t-1} + 0.149 \Delta Y_{t-1}$$
$$(t) \quad (6.969) \; (2.884) \qquad (2.587)$$

$$\Delta \hat{Y}_t = 0.006 + 0.475 \Delta C_{t-1} - 0.217 \Delta Y_{t-1}$$
$$(t) \quad (6.122) \; (4.885) \qquad (2.889)$$

- Past changes in C have positive effects on current changes in both C and Y.
- Past changes in Y have positive effects on current changes in C, but negative effect on Y.

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Impulse Responses and Variance Decompositions

- Useful tools in macroeconomics
- Analyze problems such as the effect of an oil price shock on inflation and GDP growth, and the effect of a change in monetary policy on the economy
- Impulse response functions show the effects of shocks on the adjustment path of the variables
- Variance decompositions show the effects of shocks on the forecast error variance
- A VAR model tells whether series are significantly related to each other; an impulse response analysis shows how series react dynamically to shocks; a variance decomposition analysis is informative about the sources of volatility

Impulse Responses and Variance Decompositions

- ## Impulse response function
- Consider a univariate series: $y_t = \rho y_{t-1} + v_t$
- If the series is subject to a shock of size v in period 1, the value of y in period 1 and subsequent periods will be:

$$t = 1, \ \ y_1 = \rho y_0 + v_1 = v$$

$$t = 2, \ y_2 = \rho y_1 = \rho v$$

$$t = 3, \ \ y_3 = \rho y_2 = \rho(\rho y_1) = \rho^2 v$$

...

the shock is $v, \ \rho v, \ \rho^2 v$, … …

- The time-path of y following the shock is known as the impulse response function (IRF)

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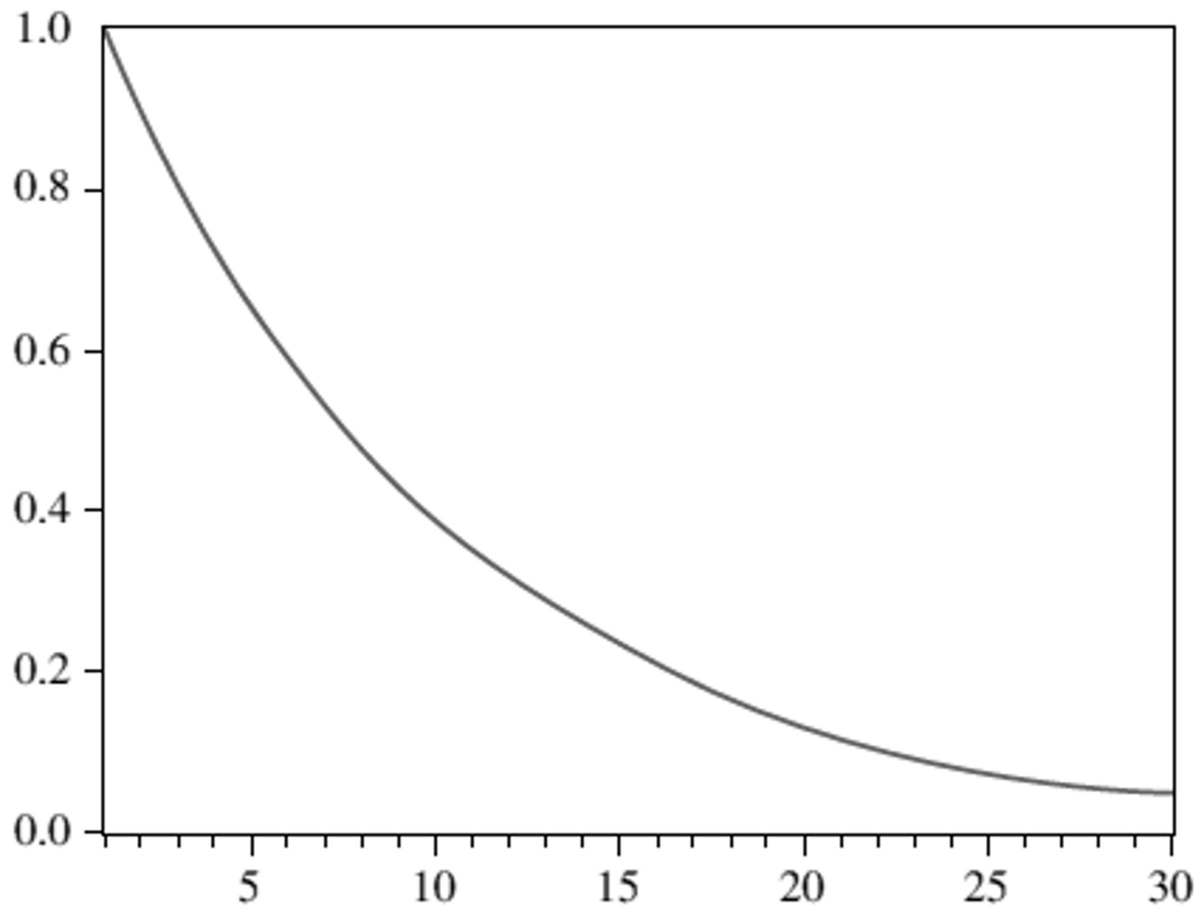
Impulse Responses and Variance Decompositions

■ Impulse response function: example



Impulse responses for an AR(1) model $y_t = 0.9y_{t-1} + e_t$ following a unit shock

Impulse Responses and Variance Decompositions

- Impulse response function
- Consider a bivariate VAR system of stationary variables:

$$y_t = \delta_{10} + \delta_{11} y_{t-1} + \delta_{12} x_{t-1} + v_t^y$$

$$x_t = \delta_{20} + \delta_{21} y_{t-1} + \delta_{22} x_{t-1} + v_t^x$$

- when there is a one–standard deviation shock (alternatively called an innovation) to ***y***:

Let $v_1^y = \sigma_y$, $v_t^y = 0$ for $t > 1$, $v_t^x = 0$ for all t:

$$t = 1 \quad y_1 = v_1^y = \sigma_y$$

$$x_1 = v_1^x = 0$$

$$t = 2 \quad y_2 = \delta_{11} y_1 + \delta_{12} x_1 = \delta_{11}\sigma_y + \delta_{12} 0 = \delta_{11}\sigma_y$$

$$x_2 = \delta_{21} y_1 + \delta_{22} x_1 = \delta_{21}\sigma_y + \delta_{22} 0 = \delta_{21}\sigma_y$$

$$t = 3 \quad y_3 = \delta_{11} y_2 + \delta_{12} x_2 = \delta_{11}\delta_{11}\sigma_y + \delta_{12}\delta_{21}\sigma_y$$

$$x_3 = \delta_{21} y_2 + \delta_{22} x_2 = \delta_{21}\delta_{11}\sigma_y + \delta_{22}\delta_{21}\sigma_y$$

...

impulse response to y on y: $\sigma_y\{1,\ \delta_{11},\ \left(\delta_{11}\delta_{11} + \delta_{12}\delta_{21}\right)\ , \ldots \ldots\}$

impulse response to y on x: $\sigma_y\{0,\ \delta_{21},\ \left(\delta_{21}\delta_{11} + \delta_{22}\delta_{21}\right)\ , \ldots \ldots\}$

Impulse Responses and Variance Decompositions

- Impulse response function
- Consider a bivariate VAR system of stationary variables:

$$y_t = \delta_{10} + \delta_{11} y_{t-1} + \delta_{12} x_{t-1} + v_t^y$$

$$x_t = \delta_{20} + \delta_{21} y_{t-1} + \delta_{22} x_{t-1} + v_t^x$$

- when there is a one–standard deviation shock (alternatively called an innovation) to $\boldsymbol{x}$:

Now let $v_1^x = \sigma_x$, $v_t^x = 0$ for $t > 1$, $v_t^y = 0$ for all t:

$$t = 1 \quad y_1 = v_1^y = 0$$

$$x_1 = v_t^x = \sigma_x$$

$$t = 2 \quad y_2 = \delta_{11} y_1 + \delta_{12} x_1 = \delta_{11} 0 + \delta_{12}\sigma_x = \delta_{12}\sigma_x$$

$$x_2 = \delta_{21} y_1 + \delta_{22} x_1 = \delta_{21} 0 + \delta_{22}\sigma_x = \delta_{22}\sigma_x$$

...

impulse response to x on y: $\sigma_x\{0,\ \delta_{12},\ \left(\delta_{11}\delta_{12} + \delta_{12}\delta_{22}\right),\ \ldots\ \ldots\}$

impulse response to x on x: $\sigma_x\{1,\ \delta_{22},\ \left(\delta_{21}\delta_{12} + \delta_{22}\delta_{22}\right),\ \ldots\ \ldots\}$

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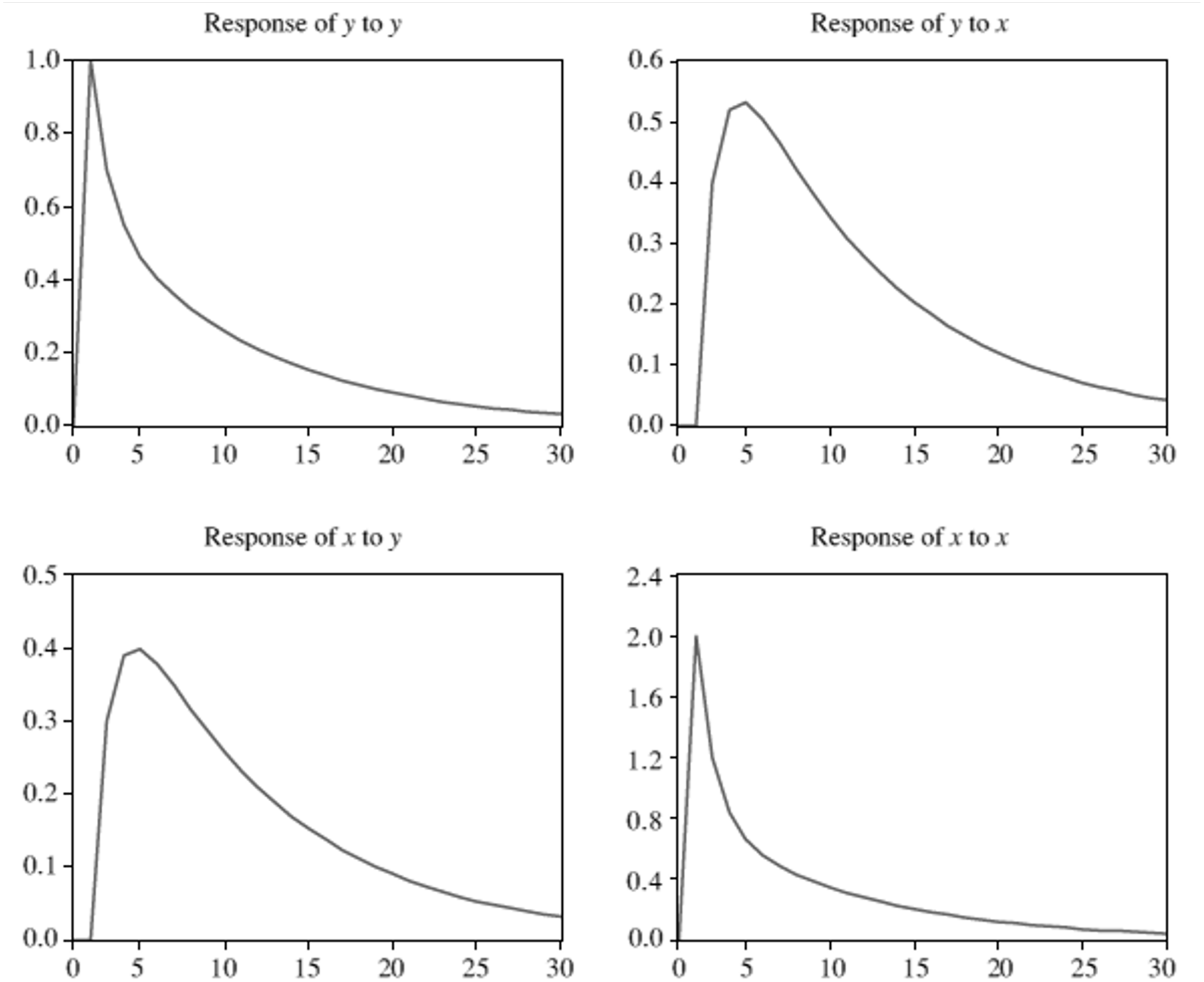
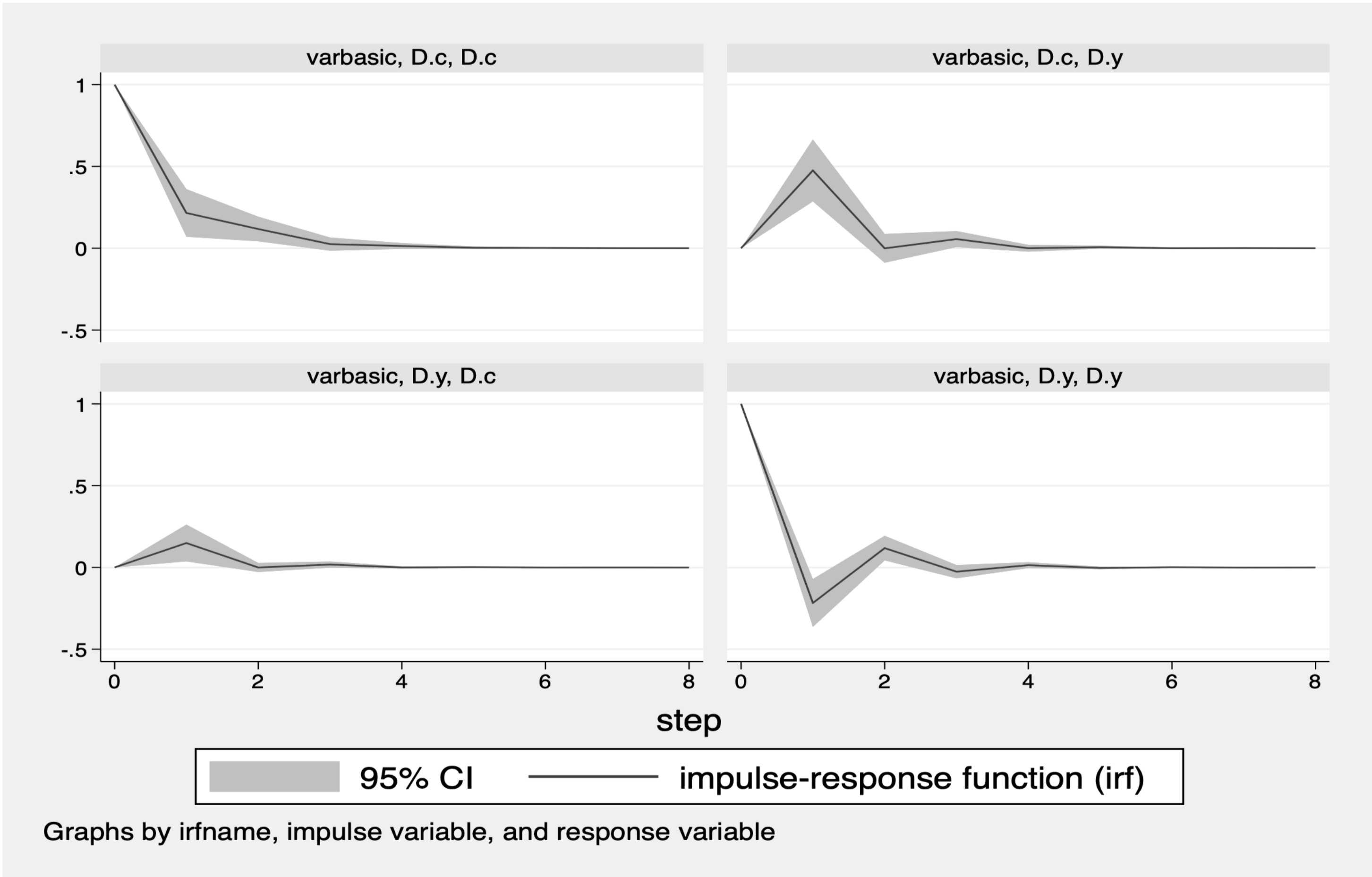
Impulse Responses and Variance Decompositions



Impulse Responses and Variance Decompositions

- Example: Real personal disposable income (Y) and real personal consumption expenditure (C)



Impulse Responses and Variance Decompositions

- Variance Decomposition: attributing the source of the variation in the forecast error
- Consider a univariate series: $y_t = \rho y_{t-1} + v_t$
- The best one-step-ahead forecast (alternatively the forecast one period ahead) is:

$$y_t = \rho y_{t-1} + v_t$$

$$y_{t+1}^F = E_t[\rho y_t + v_{t+1}] = \rho y_t$$

$$y_{t+1} - E_t[y_{t+1}] = y_{t+1} - \rho y_t = v_{t+1}$$

$$y_{t+2}^F = E_t[\rho y_{t+1} + v_{t+2}] = E_t[\rho(\rho y_t + v_{t+1}) + v_{t+2}] = \rho^2 y_t$$

$$y_{t+2} - E_t[y_{t+2}] = y_{t+2} - \rho^2 y_t = \rho v_{t+1} + v_{t+2}$$

- The forecast error variance is 100% due to its own shock

Impulse Responses and Variance Decompositions

- Variance Decomposition: attributing the source of the variation in the forecast error
- Consider a bivariate example:

$$y_t = \delta_{10} + \delta_{11} y_{t-1} + \delta_{12} x_{t-1} + v_t^y$$

$$x_t = \delta_{20} + \delta_{21} y_{t-1} + \delta_{22} x_{t-1} + v_t^x$$

- Ignoring the intercepts (since they are constants), the one–step ahead forecasts are:

$$y_{t+1}^F = E_t[\delta_{11} y_t + \delta_{12} x_t + v_{t+1}^y] = \delta_{11} y_t + \delta_{12} x_t$$

$$x_{t+1}^F = E_t[\delta_{21} y_t + \delta_{22} x_t + v_{t+1}^x] = \delta_{21} y_t + \delta_{22} x_t$$

$$FE_1^y = y_{t+1} - E_t[y_{t+1}] = v_{t+1}^y; \quad \text{var}(FE_1^y) = \sigma_y^2$$

$$FE_1^x = x_{t+1} - E_t[x_{t+1}] = v_{t+1}^x; \quad \text{var}(FE_1^x) = \sigma_x^2$$

- The forecast error variance is 100% due to its own shock

Impulse Responses and Variance Decompositions

- Variance Decomposition (bivariate case cont'd)
- The two–step ahead forecasts are:

$$y_{t+2}^{F} = E_t[\delta_{11} y_{t+1} + \delta_{12} x_{t+1} + v_{t+2}^{y}]$$

$$= E_t[\delta_{11}\left(\delta_{11} y_t + \delta_{12} x_t + v_{t+1}^{y}\right) + \delta_{12}\left(\delta_{21} y_t + \delta_{22} x_t + v_{t+1}^{x}\right) + v_{t+2}^{y}]$$

$$= \delta_{11}\left(\delta_{11} y_t + \delta_{12} x_t\right) + \delta_{12}\left(\delta_{21} y_t + \delta_{22} x_t\right)$$

$$FE_2^{y} = y_{t+2} - y_{t+2}^{F} = \delta_{11} v_{t+1}^{y} + \delta_{12} v_{t+1}^{x} + v_{t+2}^{y} \qquad \text{var}(FE_2^{y}) = \delta_{11}^2 \sigma_y^2 + \delta_{12}^2 \sigma_x^2 + \sigma_y^2$$

$$x_{t+2}^{F} = E_t[\delta_{21} y_{t+1} + \delta_{22} x_{t+1} + v_{t+2}^{x}]$$

$$= E_t[\delta_{21}(\delta_{11} y_t + \delta_{12} x_t + v_{t+1}^{y}) + \delta_{22}(\delta_{21} y_t + \delta_{22} x_t + v_{t+1}^{x}) + v_{t+2}^{x}]$$

$$= \delta_{21}(\delta_{11} y_t + \delta_{12} x_t) + \delta_{22}(\delta_{21} y_t + \delta_{22} x_t)$$

$$FE_2^{x} = y_{t+2} - y_{t+2}^{F} = \delta_{21} v_{t+1}^{y} + \delta_{22} v_{t+1}^{x} + v_{t+2}^{x} \qquad \text{var}(FE_2^{x}) = \delta_{21}^2 \sigma_y^2 + \delta_{22}^2 \sigma_x^2 + \sigma_x^2$$

Impulse Responses and Variance Decompositions

- This decomposition is often expressed in proportional terms

Variable	Forecasting error	% explained by own shock	% explained by other shock
Y	$\delta_{11}^2\sigma_y^2+\delta_{12}^2\sigma_x^2+\sigma_y^2$	$\dfrac{\delta_{11}^2\sigma_y^2+\sigma_y^2}{\delta_{11}^2\sigma_y^2+\delta_{12}^2\sigma_x^2+\sigma_y^2}$	$\dfrac{\delta_{12}^2\sigma_x^2}{\delta_{11}^2\sigma_y^2+\delta_{12}^2\sigma_x^2+\sigma_y^2}$
X	$\delta_{21}^2\sigma_y^2+\delta_{22}^2\sigma_x^2+\sigma_x^2$	$\dfrac{\delta_{22}^2\sigma_x^2+\sigma_x^2}{\delta_{21}^2\sigma_y^2+\delta_{22}^2\sigma_x^2+\sigma_x^2}$	$\dfrac{\delta_{21}^2\sigma_y^2}{\delta_{21}^2\sigma_y^2+\delta_{22}^2\sigma_x^2+\sigma_x^2}$

- Contemporaneous interactions and correlated errors complicate the identification of the nature of shocks and hence the interpretation of the impulses and decomposition of the causes of the forecast error variance

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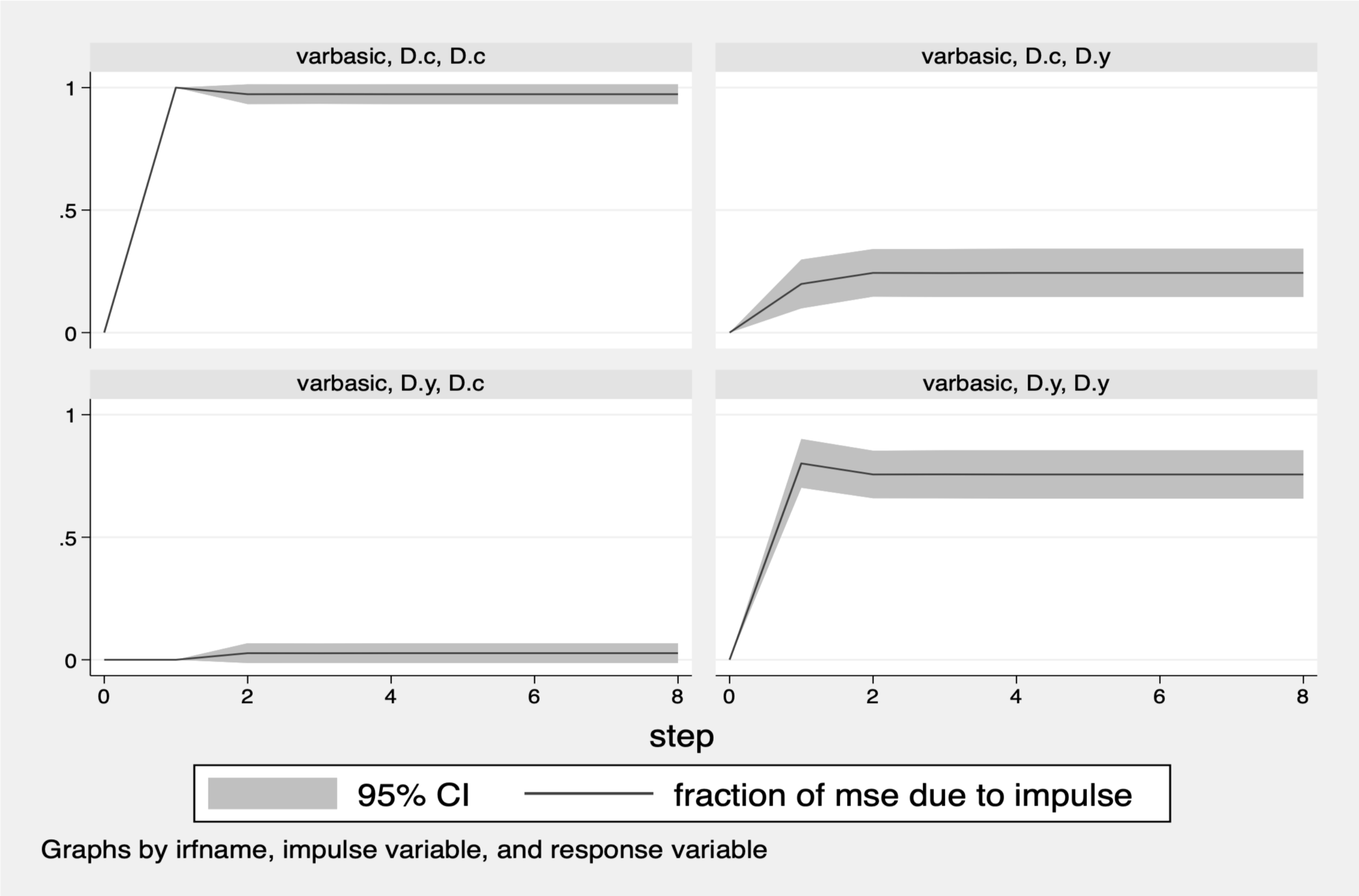
Impulse Responses and Variance Decompositions

- Example: Real personal disposable income (Y) and real personal consumption expenditure (C)



Summary

- VEC and VAR Models
- Estimating a Vector Error Correction Model
- Estimating a VAR Model
- Impulse Responses and Variance Decomposition