Topic One
Regression with Time Series Data: Stationary Variables

Outline

- Introduction
- Finite Distributed Lags
- Serial Correlation
- Estimation with Serially Correlated Errors
- Autoregressive Distributed Lag Models
- Forecasting
- Multiplier Analysis

Introduction

- When modeling relationships between variables, the nature of the data that have been collected has an important bearing on the appropriate choice of an econometric model
  - Two features of time-series data to consider:
    1. Time-series observations on a given economic unit, observed over a number of time periods, are likely to be **correlated**
    2. Time-series data have **a natural ordering** according to time

Introduction

- There is also the possible existence of dynamic relationships between variables
  - A dynamic relationship is one in which the change in a variable now has an impact on that same variable, or other variables, in **one or more future time periods**
  - These effects do not occur instantaneously but are spread, or **distributed**, over future time periods

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Introduction

- Ways to model the dynamic relationship:
  1. Specify that a dependent variable y is a function of current and past values of an explanatory variable x

$$y_t = f(x_t, x_{t-1}, x_{t-2}, \ldots)$$

  - Because of the existence of these lagged effects, this equation is called a **distributed lag (DL) model**

Introduction

- Ways to model the dynamic relationship (Continued):
  2. Capturing the dynamic characteristics of time-series by specifying a model with a lagged dependent variable as one of the explanatory variables

$$y_t = f(y_{t-1}, x_t)$$

or

$$y_t = f(y_{t-1}, x_t, x_{t-1}, x_{t-2})$$

– Such models are called **autoregressive distributed lag (ARDL)** models, with "autoregressive" meaning a regression of y_t on its own lag or lags

Introduction

- Ways to model the dynamic relationship (Continued):
  3. Model the continuing impact of change over several periods via the error term

$$y_t = f(x_t) + e_t \qquad e_t = f(e_{t-1})$$

- In this case e_t is correlated with e_{t-1}
- We say the errors are **serially correlated** or **autocorrelated**

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Introduction

- The primary assumption in OLS is:

$$\operatorname{cov}\left(y_i, y_j\right) = \operatorname{cov}\left(e_i, e_j\right) = 0 \quad \text{for } i \neq j$$

  - For time series, this is written as:

$$\operatorname{cov}\left(y_t, y_s\right) = \operatorname{cov}\left(e_t, e_s\right) = 0 \quad \text{for } t \neq s$$

    – The dynamic models mentioned previously imply correlation between y_t and y_{t-1} or e_t and e_{t-1} or both, so they clearly violate this assumption

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Finite Distributed Lags

- Consider a linear model in which, after q time periods, changes in x no longer have an impact on y

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_q x_{t-q} + e_t$$

- This model has two uses:
  – Forecasting

$$y_{T+1} = \alpha + \beta_0 x_{T+1} + \beta_1 x_T + \beta_2 x_{T-1} + \cdots + \beta_q x_{T-q+1} + e_{T+1}$$

  – Policy analysis
    • What is the effect of a change in x on y?

$$\frac{\partial E(y_t)}{\partial x_{t-s}} = \frac{\partial E(y_{t+s})}{\partial x_t} = \beta_s$$

Finite Distributed Lags

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_q x_{t-q} + e_t$$

- The effect of a one-unit change in x_t is **distributed** over the current and next q periods, from which we get the term "distributed lag model"
  - It is called a **finite distributed lag model of order *q***
    - It is assumed that after a finite number of periods q, changes in x no longer have an impact on y
  - The coefficient β_s is called a **distributed-lag weight** or an ***s*-period delay multiplier**
  - The coefficient β_0 ($s = 0$) is called the **impact multiplier**

Finite Distributed Lags

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_q x_{t-q} + e_t$$

- Assume x_t is increased by one unit and then maintained at its new level in subsequent periods
  - The immediate impact will be β_0
  - the total effect in period $t + 1$ will be $\beta_0 + \beta_1$, in period $t + 2$ it will be $\beta_0 + \beta_1 + \beta_2$, and so on
    - These quantities are called **interim multipliers**
  - The **total multiplier** is the final effect on y of the sustained increase after q or more periods have elapsed

$$\sum_{s=0}^{q} \beta_s$$

Finite Distributed Lags

- Example: Okun's Law
  - The change in the unemployment rate depends on the rate of growth of output in the economy:

$$U_t - U_{t-1} = -\gamma\left(G_t - G_N\right)$$

  - We can rewrite this as:

$$DU_t = \alpha + \beta_0 G_t + e_t$$

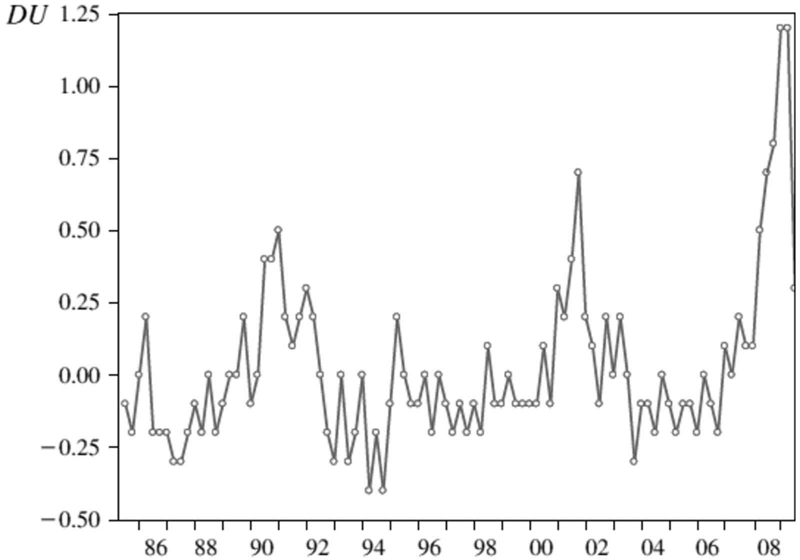
where $DU = \Delta U = U_t - U_{t-1}$, $\beta_0 = -\gamma$, $\alpha = \gamma G_N$, and

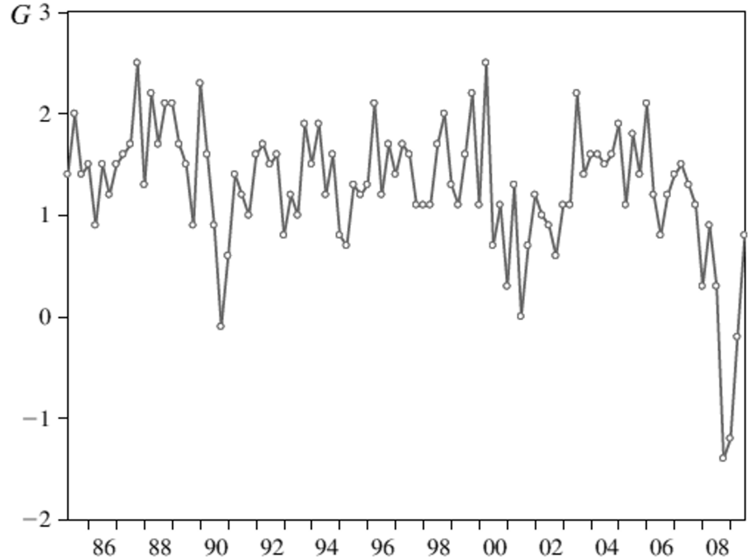
$$G_t = \frac{GDP_t - GDP_{t-1}}{GDP_{t-1}} \times 100$$

  - We can expand this to include lags:

$$DU_t = \alpha + \beta_0 G_t + \beta_1 G_{t-1} + \beta_2 G_{t-2} + \cdots + \beta_q G_{t-q} + e_t$$

Finite Distributed Lags





t	Quarter	U_t	U_{t-1}	DU_t	G_t	G_{t-1}	G_{t-2}	G_{t-3}
1	1985Q2	7.3	•	•	1.4	•	•	•
2	1985Q3	7.2	7.3	−0.1	2.0	1.4	•	•
3	1985Q4	7.0	7.2	−0.2	1.4	2.0	1.4	•
4	1986Q1	7.0	7.0	0.0	1.5	1.4	2.0	1.4
5	1986Q2	7.2	7.0	0.2	0.9	1.5	1.4	2.0
96	2009Q1	8.1	6.9	1.2	−1.2	−1.4	0.3	0.9
97	2009Q2	9.3	8.1	1.2	−0.2	−1.2	−1.4	0.3
98	2009Q3	9.6	9.3	0.3	0.8	−0.2	−1.2	−1.4

Finite Distributed Lags

Lag Length $q = 3$				
Variable	Coefficient	Std. Error	t-value	p-value
Constant	0.5810	0.0539	10.781	0.0000
G_t	−0.2021	0.0330	6.120	0.0000
G_{t-1}	−0.1645	0.0358	−4.549	0.0000
G_{t-2}	−0.0716	0.0353	−2.027	0.0456
G_{t-3}	0.0033	0.0363	0.091	0.9276
Observations = 95	$R^2 = 0.652$		$\hat{\sigma} = 0.1743$	

Lag Length $q = 2$				
Variable	Coefficient	Std. Error	t-value	p-value
Constant	0.5836	0.0472	12.360	0.0000
G_t	−0.2020	0.0324	−6.238	0.0000
G_{t-1}	−0.1653	0.0335	−4.930	0.0000
G_{t-2}	−0.0700	0.0331	−2.115	0.0371
Observations = 96	$R^2 = 0.654$		$\hat{\sigma} = 0.1726$	

Serial Correlation

- When a variable exhibits correlation over time, we say it is **autocorrelated** or **serially correlated**
- How do we test whether an autocorrelation is significantly different from zero?
  - The *k*-th order sample autocorrelation
  - Correlogram (the sample autocorrelation function)
  - Lagrange Multiplier test
  - Durbin-Watson test (used less frequently today)

Serial Correlation

- The null hypothesis is H_0: $\rho_k = 0$
- The test statistic is:

$$Z = \frac{r_k - 0}{\sqrt{1/T}} = \sqrt{T}\, r_k \sim N(0,1)$$

- The sample statistic, or the ***k*-th order sample autocorrelation** for a series *y* that gives the correlation between observations that are *k* periods apart, is:

$$r_k = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2}$$

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Serial Correlation

- Example: Okun's Rule (Cont'd)
- The first four autocorrelations are:

$$r_1 = 0.494 \quad r_2 = 0.411 \quad r_3 = 0.154 \quad r_4 = 0.200$$

- The test statistics are:

$$Z_1 = \sqrt{98} \times 0.494 = 4.89, \quad Z_2 = \sqrt{98} \times 0.414 = 4.10$$

$$Z_3 = \sqrt{98} \times 0.154 = 1.52, \quad Z_4 = \sqrt{98} \times 0.200 = 1.98$$

- Critical value: 1.98
- We conclude that G, the quarterly growth rate in U.S. GDP, exhibits significant serial correlation at lags one and two

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Serial Correlation

- The **correlogram (sample autocorrelation function)** is the sequence of autocorrelations r_1, r_2, r_3, …

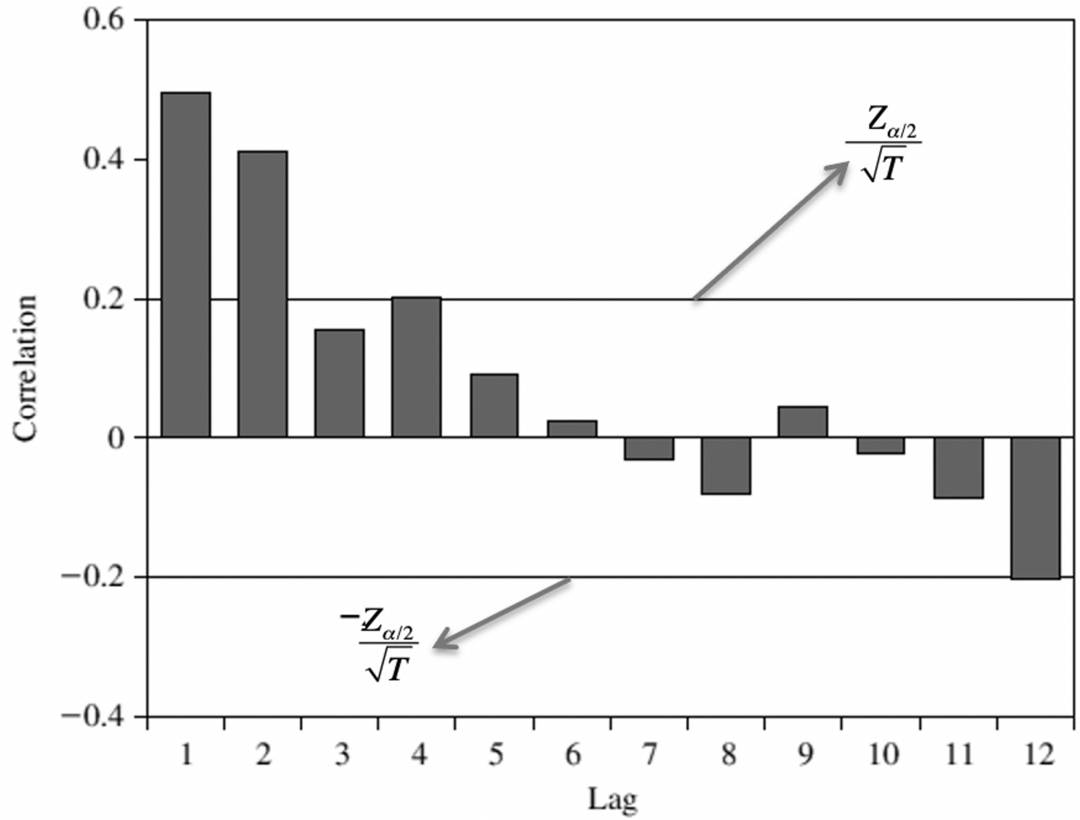

FIGURE 1.6 Correlogram for *G*

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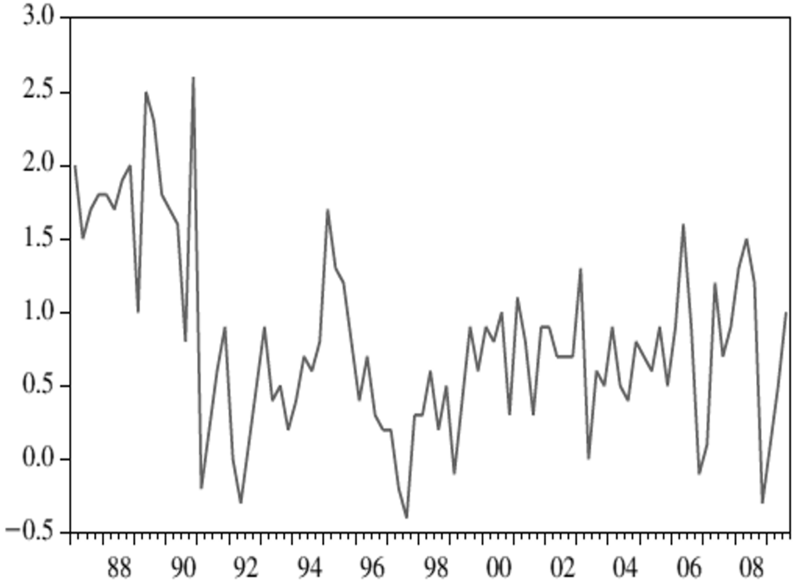
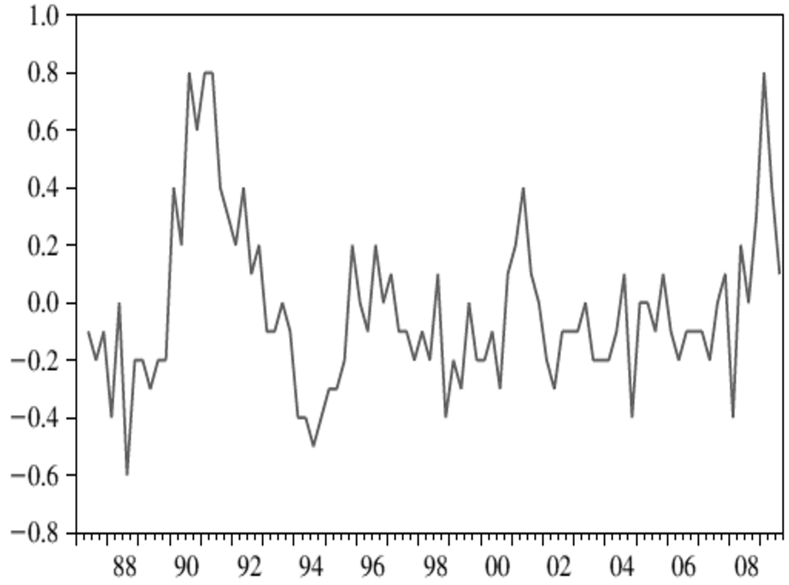
Serial Correlation

- Example: Phillips Curve (Inflation & changes of unemployment rate)

$$INF_t = \beta_1 + \beta_2 DU_t + e_t$$





Australian price inflation (left) and quarterly changes of unemployment rate (right)

Serial Correlation

- The least squares equation is:

$$\widehat{INF} = 0.7776 - 0.5279DU$$
$$(se) \quad (0.0658) \; (0.2294)$$

- The k-th order autocorrelation for the residuals can be written as:

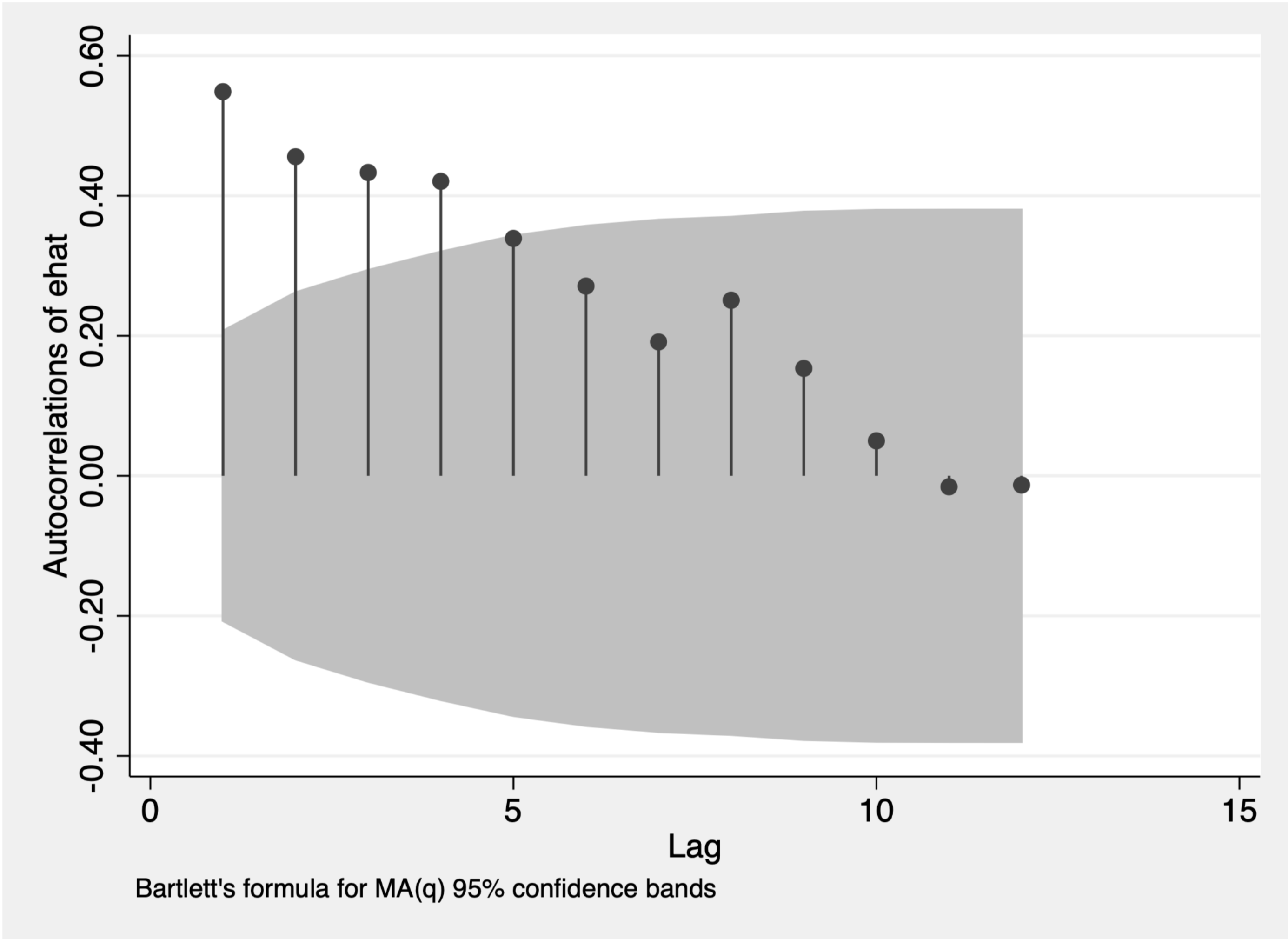
$$r_k = \frac{\sum_{t=k+1}^{T} \hat{e}_t \hat{e}_{t-k}}{\sum_{t=1}^{T} \hat{e}_t^2}$$

- The values at the first five lags are:

$$r_1 = 0.549 \quad r_2 = 0.456 \quad r_3 = 0.433 \quad r_4 = 0.420 \quad r_5 = 0.339$$

- The significance bounds are $1.96/\sqrt{90} = 0.21$

Serial Correlation



Correlogram for residuals from least-squares estimated Phillips curve

Serial Correlation

- LM Test: a joint test of correlations at more than one lag
- In a simple linear regression model

$$y_t = \beta_1 + \beta_2 x_t + e_t$$

- If residuals are correlated, then one way to model the relationship between them is to write:

$$e_t = \rho_1 e_{t-1} + \rho_2 e_{t-2} + \ldots \rho_p e_{t-p} + v_t$$

- Test H_0: $\rho = 0$ by using a linear regression equation

$$y_t = \beta_1 + \beta_2 x_t + \rho_1 \hat{e}_{t-1} + \rho_2 \hat{e}_{t-2} + \ldots \rho_p \hat{e}_{t-p} + v_t$$

$$\hat{e}_t = (\beta_1 - b_1) + (\beta_2 - b_2) x_t + \rho_1 \hat{e}_{t-1} + \rho_2 \hat{e}_{t-2} + \ldots \rho_p \hat{e}_{t-p} + v_t$$

- $T \times R^2 \sim \chi^2(p)$

Serial Correlation

- Example: Philips Curve (Cont'd)

$$\hat{e}_t = (\beta_1 - b_1) + (\beta_2 - b_2)DU_t + \rho_1\hat{e}_{t-1} + \rho_2\hat{e}_{t-2} + \ldots\rho_p\hat{e}_{t-p} + v_t$$

- The first p initial values of residuals are unknown. Two ways to handle this are:
  1. Delete the first i observation and use a total of T-p observations
  2. Set these initial values to zero and use all T observations
- To test H_0: $\rho = 0$ at $p = 1$ (critical value = 3.84)

$$(Method\ 1)\quad LM = (T-1)\times R^2 = 89\times 0.3102 = 27.61$$

$$(Method\ 2)\quad LM = T\times R^2 = 90\times 0.3066 = 27.59$$

- Reject the null hypothesis. Errors are serially correlated.

Serial Correlation

- Example: Philips Curve (Cont'd)

$$\widehat{e}_t = (\beta_1 - b_1) + (\beta_2 - b_2) DU_t + \rho_1 \widehat{e}_{t-1} + \rho_2 \widehat{e}_{t-2} + \dots \rho_p \widehat{e}_{t-p} + v_t$$

- To test H_0: $\rho = 0$ at $p = 4$ (critical value = 9.49)

$$(Method\ 1)\quad LM = (T-4) \times R^2 = 86 \times 0.3882 = 33.4$$

$$(Method\ 2)\quad LM = T \times R^2 = 90 \times 0.4075 = 36.7$$

- Reject the null hypothesis. Errors are serially correlated.

Estimation with Serially Correlated Errors

- Least squares estimation without recognizing the existence of serially correlated errors
  – The least squares estimator is still a linear **unbiased** estimator, but it is no longer best
  – The formulas for the **standard errors** usually computed for the least squares estimator are no longer correct
  – **Confidence intervals and hypothesis tests** that use these standard errors may be misleading

Estimation with Serially Correlated Errors

- Three estimation procedures are considered:
  1. Least Squares (LS) estimation with Heteroskedasticity and Autocorrelation Consistent (HAC) standard errors
  2. Nonlinear Least Squares with an AR(1) error specification $e_t = \rho e_{t-1} + v_t$
  3. Autoregressive Distributed Lag (ARDL) model

Estimation with Serially Correlated Errors

- LS with HAC standard errors:

  $y_t = \beta_1 + \beta_2 x_t + e_t,$ where e_t is serially correlated

- HAC (heteroskedasticity and autocorrelation consistent) standard errors, or Newey-West standard errors

- Analogous to the Heteroskedasticity Consistent (HC) Standard Errors

$$\widehat{\operatorname{var}_{HAC}(b_2)} = \widehat{\operatorname{var}_{HC}(b_2)} \times \hat{g}$$

$$\hat{g} = \left[1 + \frac{\sum\sum_{t \neq s} w_t w_s \operatorname{cov}(e_t, e_s)}{\sum_t w_t^2 \operatorname{var}(e_t)}\right]$$

$$\widehat{\operatorname{var}_{HC}(b_2)} = \sum_t w_t^2 \operatorname{var}(e_t)$$

$$w_t = (x_t - \bar{x}) \Big/ \sum_t (x_t - \bar{x})^2$$

Estimation with Serially Correlated Errors

- LS with HAC standard errors (Cont'd):
- Example: Philips Curve

$$\widehat{INF} = 0.7776 - 0.5279 DU$$

$$(0.0658) \quad (0.2294) \qquad \text{(LS Standard Errors)}$$

$$(0.1030) \quad (0.3127) \qquad \text{(HAC Standard Errors)}$$

- The *t* and *p*-values for testing H_0: $\beta_2 = 0$ are:

$$t = -0.5279/0.2294 = -2.301 \qquad p = 0.0238 \qquad \text{(from LS standard errors)}$$

$$t = -0.5279/0.3127 = -1.688 \qquad p = 0.0950 \qquad \text{(from HAC standard errors)}$$

- The *LS* standard errors give misleading test results

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Estimation with Serially Correlated Errors

- Nonlinear Least Squares with an AR(1) error specification:

$$e_t = \rho e_{t-1} + v_t \qquad [-1 < \rho < 1]$$

- ρ is the **first-order autocorrelation** for e. r_1 is an estimate for ρ

$$r_1 = \frac{\sum e_t e_{t-1}}{\sum e_t^2}$$

- ρ_k: represents the correlation between two errors that are k periods apart

$$E(e_t) = 0 \qquad \text{var}(e_t) = \sigma_e^2 = \frac{\sigma_v^2}{1-\rho^2} \qquad \text{cov}(e_t, e_{t-k}) = \frac{\rho^k \sigma_v^2}{1-\rho^2}, \quad k > 0$$

$$\rho_k = \text{corr}(e_t, e_{t-k}) = \frac{\text{cov}(e_t, e_{t-k})}{\sqrt{\text{var}(e_t)\,\text{var}(e_{t-k})}} = \frac{\rho^k \sigma_v^2 / (1-\rho^2)}{\sigma_v^2 / (1-\rho^2)} = \rho^k$$

Estimation with Serially Correlated Errors

- Nonlinear Least Squares with an AR(1) error specification:

$$e_t = \rho e_{t-1} + v_t \qquad [-1 < \rho < 1]$$

- The regression model can be re-arranged into

$$y_t = \beta_1 + \beta_2 x_t + \rho e_{t-1} + v_t$$

$$y_t = \beta_1(1-\rho) + \beta_2 x_t + \rho y_{t-1} - \rho\beta_2 x_{t-1} + v_t$$

- It is not a linear function of the parameters (β_1, β_2, ρ)
- Nonlinear least squares estimation of this equation is equivalent to using an iterative generalized least squares estimator called the Cochrane-Orcutt procedure

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Estimation with Serially Correlated Errors

- Nonlinear Least Squares with an AR(1) error specification:

$$e_t = \rho e_{t-1} + v_t \qquad [-1 < \rho < 1]$$

- Example: Philips Curve

$$INF_t = \beta_1(1-\rho) + \beta_2 DU_t + \rho INF_{t-1} - \rho\beta_2 DU_{t-1} + v_t$$

- Nonlinear LS estimates are

$$\widehat{INF} = 0.7609 - 0.6944 DU \qquad e_t = 0.557 e_{t-1} + v_t$$

$$(se) \quad (0.1245) \quad (0.2479) \qquad\qquad (0.090)$$

- Compared with the HAC LS estimates (given below), the nonlinear estimates are more accurate

$$\widehat{INF} = 0.7776 - 0.5279 DU$$

$$(0.0658) \quad (0.2294) \qquad (\text{LS Standard Errors})$$

$$(0.1030) \quad (0.3127) \qquad (\text{HAC Standard Errors})$$

Estimation with Serially Correlated Errors

- Autoregressive Distributed Lag (ARDL) model:
- The nonlinear LS regression model can be written into an ARDL model as follows:

$$y_t = \beta_1(1-\rho) + \beta_2 x_t + \rho y_{t-1} - \rho\beta_2 x_{t-1} + v_t$$

$$y_t = \delta + \theta_1 y_{t-1} + \delta_0 x_t + \delta_1 x_{t-1} + v_t$$

with the restriction $\delta_1 = -\theta_1\delta_0$ imposed

- This model can be estimated with a restricted LS method
- It reduces the number of parameters from four to three and makes the two equations equivalent
- As long as the assumption ($\delta_1 = -\theta_1\delta_0$) holds, LS estimation is valid. This can be checked by a Wald test.

Estimation with Serially Correlated Errors

- Autoregressive Distributed Lag (ARDL) model:
- Example: Philips Curve
- The ARDL estimation is

$$\widehat{INF}_t = 0.3336 + 0.5593 INF_{t-1} - 0.6882 DU_t + 0.3200 DU_{t-1}$$
$$(se) \quad (0.0899) \quad (0.0908) \quad (0.2575) \quad (0.2499)$$

- The Wald test statistic is 0.112, with a p-value of 0.738. Do not reject the null hypothesis $\delta_1 = -\theta_1\delta_0$. The ARDL estimation is valid.
- Since DU_{t-1} is not significant, this variable is dropped and the re-estimated model is

$$\widehat{INF}_t = 0.3548 + 0.5282 INF_{t-1} - 0.4909 DU_t$$
$$(se) \quad (0.0876) \quad (0.0851) \quad (0.1921)$$

Estimation with Serially Correlated Errors

$y_t = \beta_1 + \beta_2 x_t + e_t, \qquad e_t = \rho e_{t-1} + v_t$			
	LS Model with HAC SE	**Nonlinear LS Model**	**ARDL Model**
β_1	0.7776 (0.1030)	0.7609 (0.1245)	0.7570 (--)
β_2	-0.5279 (0.3127)	-0.6944 (0.2479)	-0.6882 (0.2575)
ρ	--	0.5570 (0.0900)	0.5593 (0.0908)
AR(1) test	No	Yes	Yes

Autoregressive Distributed Lag Models

- An autoregressive distributed lag (ARDL) model is one that contains both lagged x_t's and lagged y_t's

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \cdots + \delta_q x_{t-q} + \theta_1 y_{t-1} + \cdots + \theta_p y_{t-p} + v_t$$

- Two examples:

$$\text{ARDL}(1,1):\ \widehat{INF}_t = 0.3336 + 0.5593 INF_{t-1} - 0.6882 DU_t + 0.3200 DU_{t-1}$$

$$\text{ARDL}(1,0):\ \widehat{INF}_t = 0.3548 + 0.5282 INF_{t-1} - 0.4909 DU_t$$

- An ARDL (p,q) model can be transformed into one with only lagged x's which go back into the infinite past:

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \cdots + e_t = \alpha + \sum_{s=0}^{\infty} \beta_s x_{t-s} + e_t$$

- This model is called an **infinite distributed lag model**

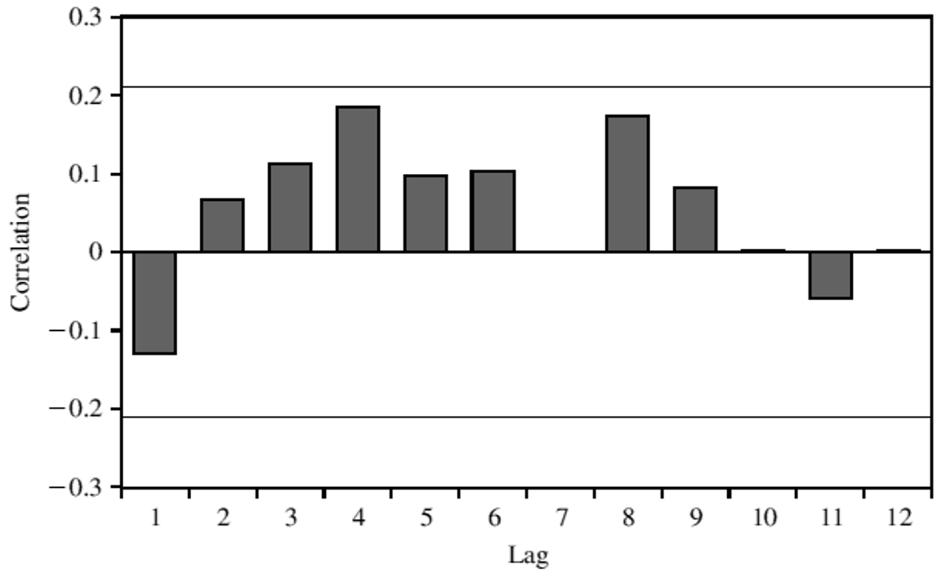
Autoregressive Distributed Lag Models

- Four possible criteria for choosing p and q:
  1. Has serial correlation in the errors been eliminated?
  2. Are the signs and magnitudes of the estimates consistent with our expectations from economic theory?
  3. Are the estimates significantly different from zero, particularly those at the longest lags?
  4. What values for p and q minimize information criteria such as the *AIC* and *SC*?

Autoregressive Distributed Lag Models

- Example: Philips Curve
- ARDL(1,0) model:

$$\widehat{INF}_t = 0.3548 + 0.5282 INF_{t-1} - 0.4909 DU_t, \quad \text{obs} = 90$$
$$(se) \quad (0.0876) \quad (0.0851) \qquad (0.1921)$$



Correlogram for residuals

Lag	*p*-value
1	0.0421
2	0.0772
3	0.1563
4	0.0486
5	0.0287

p-values for LM Test for Autocorrelation

Autoregressive Distributed Lag Models

- Example: Philips Curve : ARDL(4,0) model:

$$\widehat{INF}_t = 0.1001 + 0.2354 INF_{t-1} + 0.1213 INF_{t-2} + 0.1677 INF_{t-3}$$
$$(se) \quad (0.0983) \quad (0.1016) \quad (0.1038) \quad (0.1050)$$
$$+ 0.2819 INF_{t-4} - 0.7902 DU_t$$
$$(0.1014) \quad (0.1885) \qquad \text{obs} = 87$$

p	q	AIC	SC	p	q	AIC	SC
1	0	−1.247	−1.160	1	1	−1.242	−1.128
2	0	−1.290	−1.176	2	1	−1.286	−1.142
3	0	−1.335	−1.192	3	1	−1.323	−1.151
4	0	−1.402	−1.230	4	1	−1.380	−1.178
5	0	−1.396	−1.195	5	1	−1.373	−1.143
6	0	−1.378	−1.148	6	1	−1.354	−1.096

- Inflationary expectations are given by:

$$INF_t^E = 0.1001 + 0.2354 INF_{t-1} + 0.1213 INF_{t-2} + 0.1677 INF_{t-3} + 0.2819 INF_{t-4}$$

Autoregressive Distributed Lag Models

- Example: Okun's Law
- An ARDL (0,2) Model

$$\widehat{DU}_t = 0.5836 - 0.2020G_t - 0.1653G_{t-1} - 0.0700G_{t-2}$$
$$(se) \quad (0.0472) \quad (0.0324) \quad (0.0335) \quad (0.0331)$$

- An ARDL (1,1) Model

$$\widehat{DU}_t = 0.3780 + 0.3501DU_{t-1} - 0.1841G_t - 0.0992G_{t-1}$$
$$(se) \quad (0.0578)(0.0846) \quad (0.0307) \quad (0.0368)$$

(p, q)	AIC	SC	(p, q)	AIC	SC	(p, q)	AIC	SC
(0,1)	−3.436	−3.356	(1,1)	−3.588	−3.480	(2,1)	−3.569	−3.435
(0,2)	−3.463	−3.356	(1,2)	−3.568	−3.433	(2,2)	−3.548	−3.387
(0,3)	−3.442	−3.308	(1,3)	−3.561	−3.400	(2,3)	−3.549	−3.361

SSRMC

Autoregressive Distributed Lag Models

- An autoregressive model of order *p*, denoted AR(*p*), is given by:

$$y_t = \beta + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p} + v_t$$

- Example: Growth in real GDP

$$\widehat{G}_t = 0.4657 + 0.3770 G_{t-1} + 0.2462 G_{t-2}$$

$$(se)\ (0.1433)\quad (0.1000)\quad (0.1029)\qquad \text{obs} = 96$$

- When AIC and SC disagree, some prefer SC

Order (p)	1	2	3	4	5
AIC	−1.094	−1.131	−1.124	−1.133	−1.112
SC	−1.039	−1.049	−1.015	−0.997	−0.948

Forecasting

- AR model
- Example: Real GDP growth, an AR(2) model

$$\widehat{G}_t = 0.4657 + 0.3770G_{t-1} + 0.2462G_{t-2}$$

- The two most recent observations are

$G_{2009Q3} = 0.8$ and $G_{2009Q2} = -0.2$

- The forecast for the next three quarters are:

$$\hat{G}_{2009Q4} = 0.46573 + 0.37700 \times 0.8 + 0.24624 \times \left(-0.2\right) = 0.7181$$

$$\hat{G}_{2010Q1} = 0.4657 + 0.3770 \times 0.7181 + 0.2462 \times 0.8 = 0.9334$$

$$\hat{G}_{2010Q2} = 0.4657 + 0.3770 \times 0.9334 + 0.2462 \times 0.7181 = 0.9945$$

Forecasting

- AR model (Example: Real GDP growth)
- A 95% interval forecast for j periods into the future is given by $\hat{G}_{T+j} \pm t_{(0.975,df)} \hat{\sigma}_j$
- $\hat{\sigma}_j$ is the standard deviation of forecasting errors

$$\sigma_1^2 = \text{var}(u_1) = \sigma_v^2; \quad \sigma_2^2 = \text{var}(u_2) = \sigma_v^2 (1 + \theta_1^2)$$

$$\sigma_3^2 = \text{var}(u_3) = \sigma_v^2 \left((\theta_1^2 + \theta_2)^2 + \theta_1^2 + 1 \right)$$

Quarter	Forecast $\hat{G}_{T+j}$	Standard Error of Forecast Error ($\hat{\sigma}_j$)	Forecast Interval ($\hat{G}_{T+j} \pm 1.9858 \times \hat{\sigma}_j$)
2009Q4 ($j = 1$)	0.71808	0.55269	(−0.379, 1.816)
2010Q1 ($j = 2$)	0.93343	0.59066	(−0.239, 2.106)
2010Q2 ($j = 3$)	0.99445	0.62845	(−0.254, 2.242)

Forecasting

- ARDL (p,q) Model
- Example: Okun's Law ARDL(1,1)

$$DU_{T+1} = \delta + \theta_1 DU_T + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1}$$

- The value for G_{T+1} can be obtained from an AR model (see slides 41 & 42)
- The future value of U_t (*level*) can be estimated by

$$U_{T+1} - U_T = \delta + \theta_1 \left(U_T - U_{T-1}\right) + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1}$$

$$U_{T+1} = \delta + \left(\theta_1 + 1\right) U_T - \theta_1 U_{T-1} + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1}$$

$$= \delta + \theta_1^* U_T + \theta_2^* U_{T-1} + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1}$$

- An ARDL(1,1) model for a change in a variable can be written as an ARDL(2,1) model for the level of the same variable

Forecasting

- Exponential Smoothing
- A general form of moving average

$$\hat{y}_{T+1} = \frac{y_T + y_{T-1} + y_{T-2}}{3}$$

- If the weights decline exponentially as the observations get older:

$$\hat{y}_{T+1} = \alpha \mathrm{y}_{\mathrm{T}} + \alpha(1-\alpha)^1 y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots$$

- $0 < \alpha < 1$, and $\sum_{s=0}^{\infty} \alpha(1-\alpha)^s = 1$
- For forecasting, recognize that:

$$\hat{y}_{T+1} = \alpha y_T + (1-\alpha)\hat{y}_T$$

Forecasting

- Exponential Smoothing
- α, the smoothing parameter, reflects the relative weight of current information
- The smaller the α, the smoother the forecasting line
- It is often determined by minimizing the sum of squares of the **one-step forecast errors**

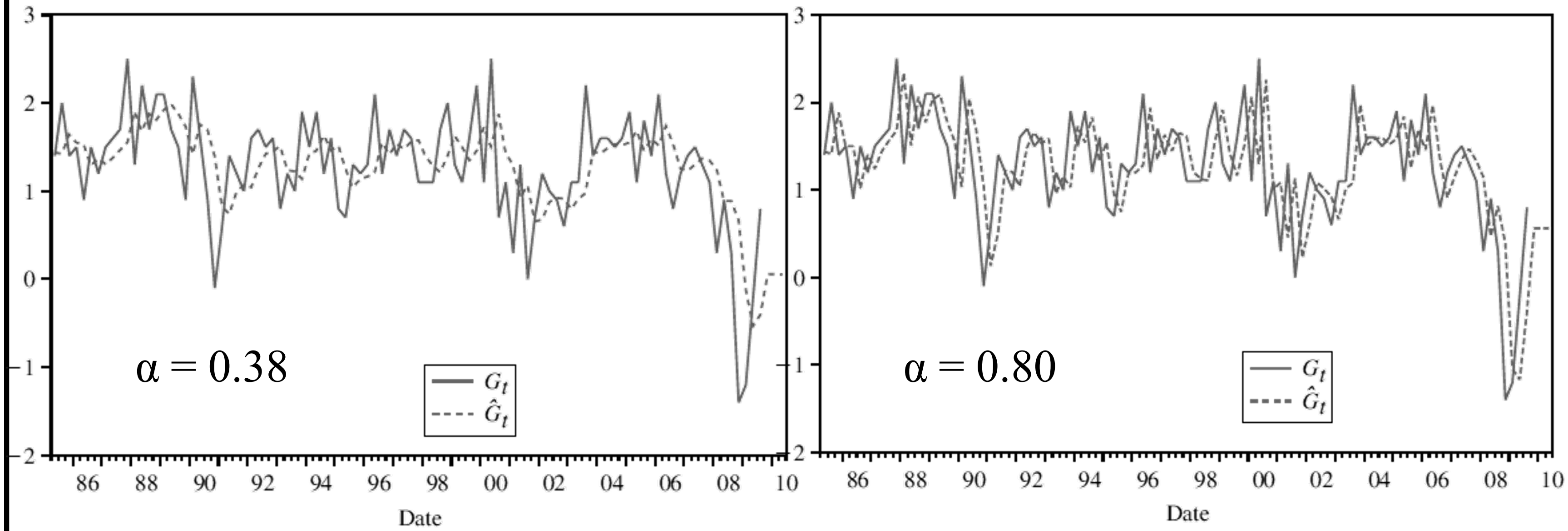
$$v_t = y_t - \hat{y}_t = y_t - \left(\alpha y_{t-1} + \left(1-\alpha\right)\hat{y}_{t-1}\right)$$

- Forecasting formula:

$$\hat{y}_t = \alpha y_{t-1} + \left(1-\alpha\right)\hat{y}_{t-1} \quad t = 2,3,\ldots,T$$

Forecasting

Exponential Smoothing



- The forecasts for 2009Q4 are:

$$\alpha = 0.38: \quad \hat{G}_{T+1} = 0.38 \times 0.8 + (1 - 0.38) \times (-0.403921) = 0.0536$$

$$\alpha = 0.8: \quad \hat{G}_{T+1} = 0.8 \times 0.8 + (1 - 0.8) \times (-0.393578) = 0.5613$$

Multiplier Analysis

- Multiplier Analysis: the effect, and the timing of the effect, of a change in one variable on the outcome of another variable
- For an ARDL model of the form:

$$y_t = \delta + \theta_1 y_{t-1} + \cdots + \theta_p y_{t-p} + \delta_0 x_t + \delta_1 x_{t-1} + \cdots + \delta_q x_{t-q} + v_t$$

- We can transform this into

$$y_t = \alpha + \beta_0 \mathrm{x}_\mathrm{t} + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \cdots + e_t$$

$$\beta_s = \frac{\partial y_t}{\partial x_{t-s}} = s \text{ period delay multiplier}$$

$$\sum_{j=0}^{s} \beta_j = s \text{ period interim multiplier}$$

$$\sum_{j=0}^{\infty} \beta_j = \text{total multiplier}$$

$$\beta_0 = \delta_0$$

$$\beta_1 = \delta_1 + \beta_0 \theta_1$$

$$\beta_j = \beta_{j-1} \theta_1 \text{ for } j \geq 2$$

$$\sum_{j=0}^{\infty} \beta_j = \hat{\delta}_0 + \frac{\hat{\delta}_1 + \hat{\delta}_0 \hat{\theta}_1}{1 - \hat{\theta}_1}$$

SSRMC

Multiplier Analysis

- Multiplier Analysis
- Example: Okun's Law, ARDL(1,1)

$$DU_t = \delta + \theta_1 DU_{t-1} + \delta_0 G_t + \delta_1 G_{t-1} + v_t$$

- The estimated model is

$$\widehat{DU}_t = 0.3780 + 0.3501 DU_{t-1} - 0.1841 G_t - 0.0992 G_{t-1}$$

- The impact and delay multipliers are

$$\hat{\beta}_0 = \hat{\delta}_0 = -0.1841$$

$$\hat{\beta}_1 = \hat{\delta}_1 + \hat{\beta}_0 \hat{\theta}_1 = -0.099155 - 0.184084 \times 0.350116 = -0.1636$$

$$\hat{\beta}_2 = \hat{\beta}_1 \hat{\theta}_1 = -0.163606 \times 0.350166 = -0.0573$$

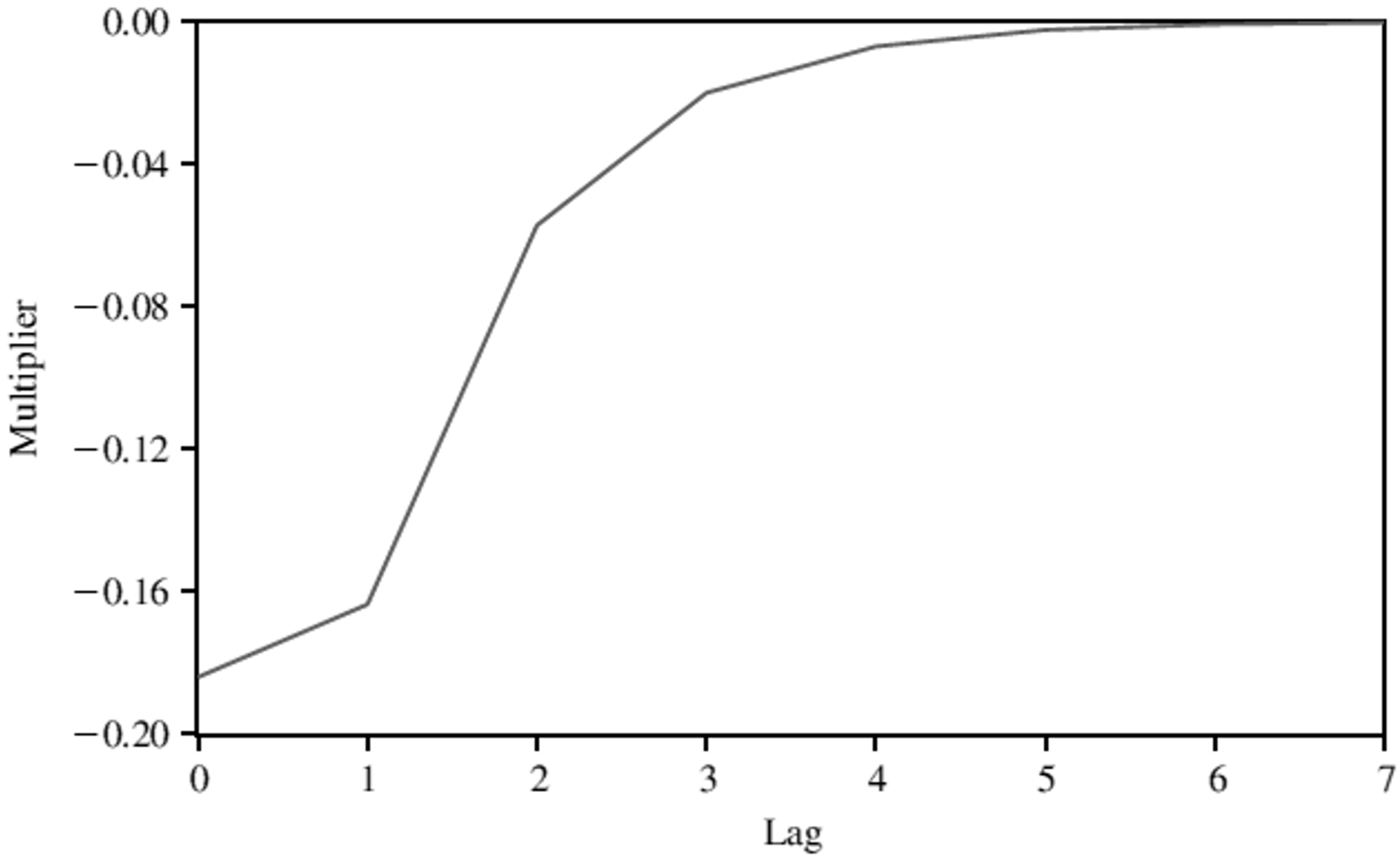
$$\hat{\beta}_3 = \hat{\beta}_2 \hat{\theta}_1 = -0.057281 \times 0.350166 = -0.0201$$

$$\hat{\beta}_4 = \hat{\beta}_3 \hat{\theta}_1 = -0.020055 \times 0.350166 = -0.0070$$

SSRMC

Multiplier Analysis

- Multiplier Analysis
- Example: Okun's Law, ARDL(1,1)



Delay multipliers from Okun's law ARDL(1,1) model

Multiplier Analysis

- Multiplier Analysis
- Example: Okun's Law, ARDL(1,1)

$$\widehat{DU}_t = 0.3780 + 0.3501 DU_{t-1} - 0.1841 G_t - 0.0992 G_{t-1}$$

- The total multiplier is

$$\sum_{j=0}^{\infty} \beta_j = \hat{\delta}_0 + \frac{\hat{\delta}_1 + \hat{\delta}_0 \hat{\theta}_1}{1 - \hat{\theta}_1} = -0.184084 + \frac{-0.163606}{1 - 0.350116} = -0.4358$$

- The normal growth rate that is needed to maintain a constant rate of unemployment:

$$G_N = -\alpha \Big/ \sum_{j=0}^{\infty} \beta_j \qquad \hat{\alpha} = \frac{\hat{\delta}}{1 - \hat{\theta}_1} = \frac{0.37801}{0.649884} = 0.5817$$

$$\hat{G}_N = \frac{0.5817}{0.4358} = 1.3\% \text{ per quarter}$$